NOTE 10

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UNIFORMLY DISTRIBUTED RANDOM DIRECTIONS IN BOUNDED SPHERICAL AREAS

Part II: Non-Conventional Methods

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Abstract In this second part the attention is first focused onto small area uniform distributions on the three dimensional sphere. These areas are limited to figures whose circumference consists of small and great circle arcs for which uniform distributions cannot directly be obtained by the usual formulation of definite integrals over polar co-ordinates. To generate the distributions a novel method is introduced which is based on analytical area ratios. Applications comprise first of all ratios of areas contained inside great circle arcs and are further complemented by figures in which some or all of the boundaries are small circle arcs. It will be shown in the case of a spherical triangle, that the surface ratio method is not necessarily more effective than the toss away alternatives one can construct with conventional random direction generation methods introduced in part I. However, for areas involving a small circle in their circumference, a simple example is given which cannot directly be solved by means of conventional tools. In the second half of this note another novel mathematical uniform random direction generation method is proposed which is also applicable in higher dimensions but limited to spherical rectangles bounded by the equivalent of small circle arcs in planes orthogonal to the mathematical equator. This method relies on the introduction of generalized meridians which themselves are at the origin of an upper limit restriction to be imposed on the feasible rectangle size.

Introduction

In this note we look at effective random direction generation inside given boundaries in three dimensions avoiding any toss away intervention. One knows that this is equivalent to the generation of random points inside figures on the S^2 sphere. We will no longer look at the Rodrigues four-vector addressed in our Note 8.

In a first time we address trigonometrical figures, where the adjective 'trigonometrical' also includes arcs of well defined small circles. It is understood that the definition of such a small circle arc also requires the knowledge of its corresponding circle center. As in Note 8, the methods limited to S^2 further rely on right ascension and colatitude co-ordinates where the adequately selected pole location corresponds to the z-axis of some 'ad hoc' orthogonal Cartesian co-ordinate system. Employing these polar co-ordinates allows to construct judiciously selected area ratios, as functions of the right ascension α. These areas are equal to the cumulative distribution function $F_\alpha(x)$. But $F_\alpha(x)$ could alternatively obtained by integrating the probability density function $p(\alpha)$, abbreviated by pdf, up to the corresponding intermediate value α_0 inside the constraint interval applicable to the

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geometrical figure considered. Employing analytical areas thus allows to avoid the need to explicitly construct a pdf, because one gets its actual integration without intervention of calculus.

In a second part we introduce the mathematical background – relying this time on Cartesian co-ordinates – which almost directly leads to the generation of uniformly distributed directions on squares or rectangles on S^2 and even on spheres in higher dimensions. To guaranty the statistical uniformity, these figures are subject to a size constraint which is broad enough to allow correct random direction generation is most practical applications.

AREA RATIO ALGORITHMS

In this section we deal with analytical area descriptions for geometrical figures whose boundaries comprise small and great circle arcs. The latter are the subject of conventional spherical trigonometry for which the area formula is well known. Combining sectors of spherical caps with canonical spherical triangles allows to derive the analytical surface of non-canonical trigonometrical figures without problems as we will show in the last subsection hereafter. By non-canonical we mean figures involving arcs of small circles.

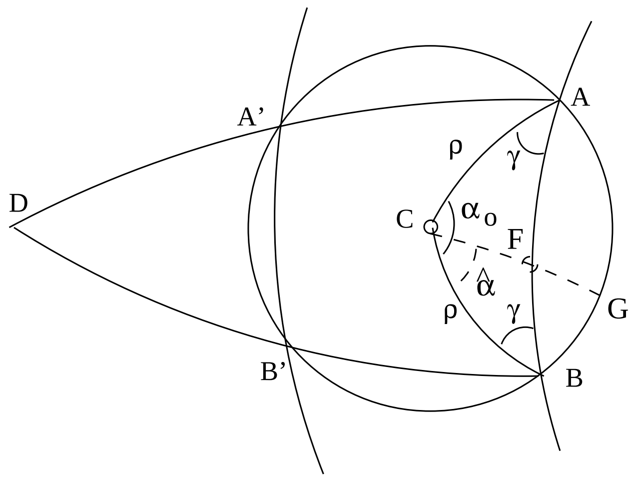

Fig. 1 Small and great circle arcs

Before proceeding, we have to extend the notational conventions for arcs and enclosed areas by referring to figure 1. In this figure we show a small circle with angular radius $\rho < \pi/2$ centered at C and intersecting four great circles giving rise to the arcs AA', BB', AB and A'B'. But the end points of these arcs are also the end point of small circle arcs which we identify by writing $\mathrm{A^sA'^s}, \mathrm{B^sB'^s}, \mathrm{A^sB^s}, \mathrm{A'^sB'^s}$ and so we equally write $\mathcal{T}(\mathrm{DA^sB^s})$ to denote the improper triangle comprising the arc $\mathrm{A^sB^s}$ in its circumference. We further call the area enclosed by AB and $\mathrm{A^sB^s}$ a (small circle) segment and $\mathcal{T}(\mathrm{CA^sB^s})$ a (small circle or spherical cap) sector whose surface is known to be $\mathcal{S}(\mathrm{CA^sB^s}) = \alpha_0\,(1 - \cos\rho)$ relying on the notations in Fig. 1.

We note that the arcs DA and DB, both intersecting a single small circle, always lead to a concave improper triangle $\mathcal{T}(\mathrm{DA'^sB'^s})$ and a convex improper triangle $\mathcal{T}(\mathrm{DA^sB^s})$ as far as D is outside the small circle. In practical applications involving small circles it is therefore indicated to start from a vectorial figure description which satisfies the requirement to define a small circle arc by three points on the unit sphere. Here these points are the center of the small circle (C) and the end points of the arc (A^s and B^s), which yield an isosceles triangle, when connected to D by great circle arcs. To establish an easy and short link between geometrical figures bounded by great circles only, we also introduce the following conventions. Looking at figure 2 hereafter, as example, we identify a dihedral angle by $a_1a_2a_3 = a_1a_2a_i = \beta_2$, keeping in mind that dihedral angles only have a defined mathematical meaning if they apply to corner points between two great circle arcs. Further, writing a_ma_n is equivalent to the arc ϵ_{mn}.

Arbitrarily Specified Spherical Triangles

With the intention to make a performance comparison, we introduce the arbitrary spherical triangle with the same reference points as we have described in Note 8 in the context of a toss away application. The random direction generation algorithm worked out here allows the application to any spherical polygon, whose sides are great circle arcs, because such polygons can be subdivided by means of triangulation. We thus consider the three arbitrarily located unit vectors $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$ which are the corner points of a spherical triangle, but with the restriction that none of the arcs separating any two of the three directions is allowed to be larger than π. Also here, it is our strategy to perform all computations trigonometrically with the assumption that the triangle has implicitly been located so, that $\mathbf{t_1}$ coincides with the polar axis. Consequently, two of the sides of the triangle then coincide with meridians and the spherical triangle is thus completely embedded in a lune. Each time a random point has been obtained in the triangle, this has to be transformed into a direction employing the direction cosines connecting the random point to the corner points which, in fact, correspond to the triad $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$. The details are explained in Note 8.

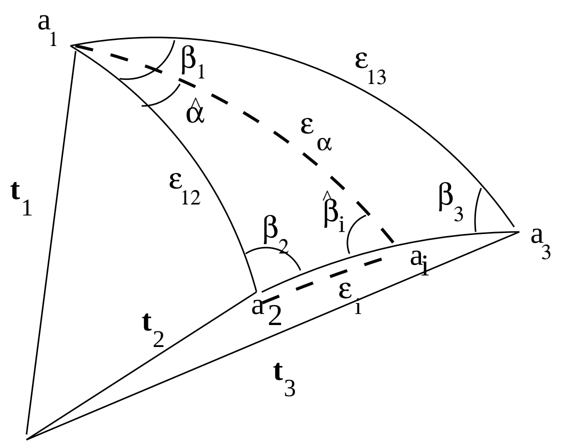

Fig. 2 Geometrical specification of the spherical triangle

Let us start by deriving all angles of the spherical triangle from the three unit vectors, shown in Fig. 2. The three arcs $a_ia_j = \epsilon_{ij}$ and the dihedral angles $a_ja_ia_k = \beta_i$ at the corners of the triangle are found from:

$$\cos\epsilon_{jk} = (\mathbf{t}_j \cdot \mathbf{t}_k) \quad \text{and} \quad \cos\epsilon_{jk} = \cos\epsilon_{ij}\,\cos\epsilon_{ik} + \sin\epsilon_{ij}\,\sin\epsilon_{ik}\cos\beta_i \tag{1a}$$

where the indices i, j, k are a permutation of 1,2,3. We will also make use of the cosine rule for dihedral angles and the sine rule of spherical trigonometry, namely

$$\cos\beta_i = -\cos\beta_j \cos\beta_k + \sin\beta_j \sin\beta_k \cos\epsilon_{jk} \quad (1b) \qquad \frac{\sin\epsilon_{ij}}{\sin\beta_k} = \text{constant} \quad (1c)$$

Once all these angles are available, we can forget for a little while the Cartesian reference provided by $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$. This is because the cosine and sine rules do not depend on the actual Cartesian reference the triangle is defined in. Hence, it is allowed to assume that we deal with a polar co-ordinate system in which a_1 coincides with the pole and thereby defines colatitude. Further, the meridian passing though a_1 and a_2 is selected to be right ascension $\alpha = 0$ reference.

Let $A(\alpha_i)$ represent the area $\mathcal{S}(\mathrm{a}_1\mathrm{a}_2\mathrm{a}_\mathrm{i})$ of the triangle with $0 \le \alpha_i \le \beta_1$. With this notation we represent the well known area of the total triangle by

$$A(\beta_1) = \beta_1 + \beta_2 + \beta_3 - \pi \tag{2}$$

And if we represent the dihedral angle at a_1 in $\mathcal{T}(\mathrm{a}_1\mathrm{a}_2\mathrm{a}_\mathrm{i})$ by $\hat{\alpha}$, the corresponding area $A(\hat{\alpha}) = \mathcal{S}(a_1 a_2 a_i)$ not only varies with the angle $\hat{\alpha}$, but also with the dihedral angle $\hat{\beta}_i = a_2 a_i a_1$. The latter angle can be computed by means of the cosine rule for dihedral angles (1b), yielding:

$$\cos\hat{\beta}_i = -\cos\hat{\alpha}\,\cos\beta_2 + \sin\hat{\alpha}\,\sin\beta_2\,\cos\epsilon_{12} \tag{3}$$

while the sine rule (1c) allows to derive $a_1 a_i = \epsilon_\alpha$ by employing:

$$\sin\epsilon_\alpha\,\sin\hat{\beta}_i = \sin\beta_2\,\sin\epsilon_{12} \tag{4}$$

Hence, the area $A(\hat{\alpha})$ can analytically be written as a function of $\hat{\alpha}$ as follows:

$$A(\hat{\alpha}) = \hat{\alpha} + \beta_2 + \arccos(-\cos\hat{\alpha}\,\cos\beta_2 + \sin\hat{\alpha}\,\sin\beta_2\,\cos\epsilon_{12}) - \pi \tag{5}$$

By closer inspection, we see that (5) divided by the total area $A(\beta_1)$, happens to be the result of the integral of the (unknown) pdf or the cumulative distribution of α from zero to $\hat{\alpha}$. To come to a uniform randomly distributed value α_t we have to solve:

$$\xi = \frac{A(\hat{\alpha})}{A(\beta_1)} = F_\alpha(\hat{\alpha}) = \int_0^{\hat{\alpha}} p(x)\,dx \tag{6}$$

for $\hat{\alpha}$ starting from a given uniform random number ξ between zero and one. Having α_t, the equations (3) and (4) provide the value of ϵ_α, which corresponds to the upper limit of ϵ at the right ascension α_t inside the triangle. Knowing this, the the random colatitude ϵ_t is obtained in the usual way as was shown in the extended Tashiro algorithm and now corresponds to step $f4$ of the algorithm hereafter.

It is suggested to perform the preparatory computations outside the direction generation algorithm. By these computations we mean all angles and arcs which remain invariable and are required for each direction generation again. Using the conventions of Note 8, we can summarize the f-triangle algorithm as follows:

f1. $\xi_1 \;=\; \rho_u^{(1)}$ with $0 \le \xi_1 \le 1$
f2. get α_t by solving (6) employing ξ_1
f3. compute ϵ_α by using α_t, (3) and (4)
f4. $\cos\epsilon_t \;=\; \xi_2 \;=\; \rho_u^{(2)}$ with $\cos\epsilon_\alpha \le \xi_2 \le 1$
f5. compute the three direction cosines with respect to the triangle corners for the direction obtained (see note 8)
f6. transform the point D with right ascension α_t and colatitude ϵ_t to a unit vector correctly located inside the triad $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ using the direction cosines of D.

A Performance test for Spherical Triangles

One will have noticed in the theoretical presentation of the area ratio method just given, that one could obtain by means of differentiation the pdf for the right ascension independently of colatitude, but the opposite is not true. There is thus the possibility of a sequential separation of probability densities, but the formal mathematical factorization in two independent pdf functions seems to be impossible. Consequently, a direct computation of expectations can, so far, not be proposed to support a performance verification.

In order to check the uniformity of a sample of random directions distributed in a spherical triangle, we propose to only rely on simple density verifications in selected fractions of the triangle. We suggest three tests consisting each of dividing the triangle in two 'test triangles' by bisecting in turn each of its dihedral angles. To explain the procedure, we bisect the dihedral angle β_1. The part containing the known arc ϵ_{12} and its two known adjacent dihedral angles $\beta_1/2$ and β_2 allow the computation of the third dihedral angle κ_1 by (1b) as follows:

$$\cos\kappa_1 \;=\; -\cos(0.5\,\beta_1)\,\cos\beta_2 \;+\; \sin(0.5\,\beta_1)\,\sin\beta_2\,\cos\epsilon_{12} \tag{7}$$

Only directions of the triangle for which $\alpha < 0.5\beta_1$ have to be counted. In this way we have all data which allow counting of directions and computing densities for each of the two triangles. If we have produced a sample of n_{tot} trial points which must all be uniformly distributed in the triangle, we should ideally sample

$$n_1 \;=\; \frac{0.5\,\beta_1 \,+\, \beta_2 \,+\, \kappa_1 \,-\, \pi}{\beta_1 \,+\, \beta_2 \,+\, \beta_3 \,-\, \pi}\; n_{tot} \tag{8}$$

points in the triangle fraction for which $\alpha \le 0.5\,\beta_1$ applies. We repeat the same procedure for the dihedral angle β_2 and β_3. In practice we will not obtain the ideal value given in (8) but the true count n_{true} which will vary from one sample to the next. Due to today's electronic computation power, we are in the comfortable situation that we can obtain the statistical results for very large samples without practical limitations. Thereby error percentages should decrease inversely proportional to the number of trials in a sample and that is what we verify in the next subsection.

A numerical example

We start from the corner points put at the right ascension and colatitude of a unit vector triad as follows $\mathbf{t_1}(10.0^0, 90.0^0)$, $\mathbf{t_2}(18.0^0, 70.0^0)$, $\mathbf{t_3}(20.0^0, 85.0^0)$. The first corner point is thus located on the equator as is shown in Fig. 3. Because the trigonometrical handling of this triangle is independent of the target location on a specified sphere we can for the toss away application as well as for the area ratio method select $\mathbf{t_1}$ to be the pole a_1. In that case the definite integral for the toss away application will extend over the improper triangle $\mathcal{T}(\mathrm{a}_2^s\mathrm{a}_0^s\mathrm{a}_1)$ and not over $\mathcal{T}(\mathrm{a}_2\mathrm{a}_0\mathrm{a}_1)$. Consequently, this covers the area of the sector of a polar cap with radius $\epsilon_{12} = a_1a_2 = a_1a_0$, whose area is equal to $\beta_1\,(1 - \cos\epsilon_{12})\mathrm{rad}^2$. The ratio of the original triangle area $\mathcal{T}(\mathrm{a}_1\mathrm{a}_2\mathrm{a}_3)$ to the area of the polar cap sector just described, is equal to 0.4802. This is by definition the geometrical efficiency of the toss away algorithm corresponding to the of the pole location chosen before. By selecting instead $\mathbf{t_2}$ this efficiency becomes 0.6777 and putting the pole at $\mathbf{t_3}$ yields the best geometrical efficiency, namely 0.68334. The geometrical efficiency of the area ratio method is not affected by this choice.

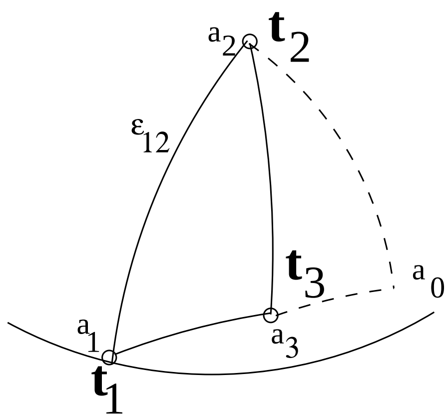

Fig. 3 Test triangle

Nevertheless, in the case of a spherical triangle, the area ratio method is numerically slightly more complex than the toss away method. In the area ratio approach the computation of a random right ascension always involves the determination a single root of a non-linear equation. For that purpose we apply the method of the chord with some 10 simple iteration steps in the mean, depending on the accuracy we want to achieve. In contrast, the toss away method relies on a direct simulation of the random right ascension. Therefore, we penalize the area ratio method by assuming the requirement of a fictitious third random number generation per direction simulation. Hence its numerical efficiency drops to 2/3=0.6667 while the toss away method is 100% numerically effective. Consequently, combining the numerical and geometrical efficiency both methods appear to be equally effective except if we would select $\mathbf{t_1}$ as working pole for the toss away method.

If we now perform the test of the methods by means of subdividing the triangle in two test triangles by bisecting dihedral angles, the question arises: "How large are the errors one may expect as a function of the sample sizes?". To this end we have added table 1. In this table we display the counts obtained for sample sizes of 100, 1000, 10,000 and 100,000 trials. In the left column we show the ratio of the test triangle area over the total triangle area for bisections at the three successive corners. The results for the toss away method

Table 1 COUNT ERRORS IN THE TEST TRIANGLES

AREA RATIO	TRIAL COUNT	100 ERR%	TRIAL COUNT	1000 ERR%	TRIAL COUNT	10,000 ERR%	TRIAL COUNT	100,000 ERR%
0.65755	61	-7.2	643	-2.2	6623	0.7	65757	0.0
	67	1.9	663	0.8	6633	0.9	65805	0.1
0.41373	41	-0.9	405	-2.1	4161	0.6	41200	0.4
	40	-3.3	392	-5.3	4022	-2.8	41353	0.0
0.42444	40	-5.8	470	10.7	4227	0.4	42534	0.2
	43	1.3	426	0.4	4264	0.5	42415	-0.1

are shown first; this is in the line containing the aforementioned ratio. A glance at the table reveals that already for samples of 100 to 1000 trials the trend is recognizable with count discrepancies up to 11%. Refined results can be expected for samples from 10,000 trials onwards. As said before, this is no problem in the light of today's computer power. We further notice that the results of both methods appear to be statistically comparable.

The Segment of a Small circle

We now describe the algorithm necessary to generate uniformly distributed random direction in a figure whose circumference consists of only an arc of a great circle and an arc of a small circle which we called a small circle segment before. For this we refer to Fig. 1. Exactly as for the arbitrary spherical triangle before, we assume that the points A, B and C in Fig. 1 correspond to vector definitions which tell us where the segment $\mathrm{FB^sG^sA^s}$ is located on the unit sphere. We then apply our geometric considerations as if $\mathcal{T}(\mathrm{CA^sB^s})$ has its point C located at the pole. Similar to the previous cases, (6) is to be replaced by

$$\xi = \frac{A(\hat{\alpha})}{A(\alpha_0)} = \frac{\mathcal{S}[\mathrm{FB^sG^s}(\hat{\alpha})]}{\mathcal{S}(\mathrm{FB^sG^sA^s})} = \frac{\mathcal{S}[\mathrm{CB^sG^s}(\hat{\alpha})] - \mathcal{S}[\mathrm{CBF}(\hat{\alpha})]}{\mathcal{S}(\mathrm{CB^sG^sA^s}) - \mathcal{S}(\mathrm{CBA})} \tag{9}$$

where ξ is the usual uniformly distributed random number comprised between zero and one. The only quantity we miss to be able to compute $\mathcal{S}[\mathrm{CBF}(\hat{\alpha})]$ is the dihedral angle $\hat{\phi}$ =CFB which we find by applying (1b). This yields

$$\hat{\phi}(\hat{\alpha}) = \arccos(-\cos\hat{\alpha}\,\cos\gamma + \sin\hat{\alpha}\,\sin\gamma\,\cos\rho) \tag{10}$$

which leads to

$$\xi = \frac{\pi - \hat{\alpha}\cos\rho - \gamma - \hat{\phi}(\hat{\alpha})}{\pi - 2\gamma - \alpha_0\,\cos\rho} \tag{11}$$

By solving this equation for $\hat{\alpha}$ we obtain the random right ascension α_t with respect to the arc CB which corresponds to $\alpha = 0$. We still need the random colatitude ϵ_t in the interval defined by $\mathrm{CF}(\alpha) \leq \epsilon_t \leq \mathrm{CG} = \rho$. Based on (1b) we derive CF from

$$\cos\gamma = -\cos\alpha\,\cos\phi + \sin\alpha\,\sin\phi\,\cos CF \tag{12}$$

Starting from $\cos\gamma$ and a further random number we derive a random colatitude ϵ_t on the meridian starting from the pole C and ending between the arcs AB and A^sB^s by applying Tashiro's method.

Thereby all the elements belonging to the generation of a random point D inside the segment of a small circle are available, but we still need the direction cosines to turn this point into a direction. The direction cosine $\cos\zeta_C$ of D with respect to C is simply $\cos\alpha_t$: With respect to B we get

$$\cos\zeta_B \;=\; \cos\rho\,\cos\epsilon_t \;+\; \sin\rho\,\sin\epsilon_t\,\cos\alpha_t \tag{13}$$

and for $\cos\zeta_A$ we replace α_t by $\alpha_0 - \alpha_t$ in the right hand side member of (13). The g-circle segment algorithm can now be summarized as follows .

g1. $\xi_1 = \rho_u^1$ with $0 \le \xi \le 1$

g2. solve (11) for $\hat{\alpha}$ using ξ_1 and obtain α_t

g3. obtain ϵ_{min} = CF from (12) using α

g4. $\cos\epsilon_t = \xi_2 = \rho_u^2$ with $\cos\rho \le \xi \le \cos\epsilon_{min}$

g5. obtain the direction cosines of the point $D(\alpha_t, \epsilon_t)$ with respect to the triad points A, B and C.

g6. compute the correctly located unit vector corresponding to D inside the unit vector triad $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$.

It must be noted that the convex surface, corresponding to the intersection of two small circles, is equal to two segments glued together along the great circle connecting the intersection points. For both circles do not need to have equal radii, the area of the segments may be different. At any rate, if one uses the algorithm just described, a random number has to decide on which side of the internal boundary the next random number has to be generated and correspondingly which sector parameters apply. If the two intersecting small circles build a concave surface, the small circle segment algorithm described before can be combined with a toss away criterion. Caveat! The distribution will be incorrect if one does not restart from scratch each time the colatitude found is not inside the prescribed boundaries. It is absolutely necessary to *also* redetermine a random right ascension if a colatitude value has been tossed away. Otherwise we introduce illegal statistical correlations.

If we wish to verify the performance of the circle segment algorithm we first remark that the great circle connecting A and B in Fig. 1 is unique. However, there is an infinity of different small circle arcs which can pass through the points A and B. Therefore, we can again use the idea of partitioning the segment in two areas. To this aim it will be sufficient to introduce an adequate auxiliary small circle arc passing through A and B so that this arc is fully inside the segment and subdivides it in two areas. To achieve this we have to move the auxiliary small circle center C' on a great circle arc through C which is perpendicular to the arc AB. To generate a small circle arc inside the segment, C' has to be moved away from the segment such the the new radius ρ' satisfies $\rho < \rho' < \pi/2$.

PSEUDO SURFACE CO-ORDINATE METHODS

. The purpose of this section is to provide a straight forward random direction generation method for spherical rectangles. By rectangles we mean figures which are defined by constraining the absolute value of the Cartesian components of a direction allowing different cut off constraints on different orthogonal co-ordinate axes. We rely on the convention that we have a vector space V_k with orthogonal base vectors along k Cartesian Cartesian co-ordinate axes. The $+x_k$-co-ordinate axis corresponds to the polar axis along the unit vector $\mathbf{v}_0$.

The generation of uniformly distributed direction in rectangles on S^2 by the present method yields figures contained inside concave small circle boundaries. These geometrical concepts loose their meaning in higher dimensions, where the method can equally be applied. In that case we propose that the name 'rectangle' is an abstraction not representable in our geometrical imagination. A very basic introduction to the higher dimensional geometry has been given in our Note 4. Hereafter we recall a few basic notions.

Almost all our geometrical considerations derive from the definition of a meridian on a sphere with a pole in dimension 3 or higher. Let $\mathbf{v}^{\perp}$ be a unit vector such that $\mathbf{v}^{\perp} \cdot \mathbf{v}_0 = 0$, then the locus of all unit vectors $\mathbf{v} = a\mathbf{v}^{\perp} + b\mathbf{v}_0$ for any a and b with $a^2 + b^2 = 1$ is called a **meridian** through the pole $\mathbf{v}_0$. This is, in fact, an alternative to 'the tangent normal decomposition' which is described by Mardia and Jupp[1]

In the k-dimensional Euclidean vector space V_k there are $(k-1)$ mutually orthogonal meridians at a pole on S^{k-1} and they only meet again at the anti pole where they are, of course, again orthogonal. We can thus introduce the Cartesian co-ordinates $(x_1, \ldots, x_k)$ as laying in the direction of the orthonormal basis $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k) \in V_k$ with $\mathbf{v}_0 = \mathbf{u}_k$. Hence, any point on a **base meridian** $\mathcal{M}_j$ generated by $\mathbf{u}_j$ and $\mathbf{v}_0$ projects on the x_j and x_k-axis only. Consistent with the terminology just proposed we will call **equator** of the sphere S^{k-1}, that part of the subspace orthogonal to $\mathbf{u}_k$ which is fully contained within the sphere S^{k-1}. Hence, the locus of all unit vectors in the equator plane is equivalent to S^{k-2}.

The meridians are a means to define the symmetry properties of a random direction distribution. This again is explained in Note 4. Rotational and axial symmetry are very well known symmetry concepts but **Rotational Uniformity** (RU) around a symmetry axis is probably not. We say that a pdf is RU if it expresses that the probability to find a random direction in the neighborhood of a meridian is equal for every meridian independent of the probability distribution inside this neighborhood. This may best be explained by an example of an error distribution which is assumed to be axially symmetric around a polar axis (although this is not a necessary condition for having a RU) . If the corresponding pdf is RU it is equally probable to find random directions near meridians where large errors are more probable as well as near meridians where small errors are more likely. A uniform random distribution of directions in a spherical square is typically not RU, because one can expect more random points along the meridians along the diagonals than along meridians parallel to the sides. The RU concept is useful, because it may point to the smaller/larger probability to be in (un)favorable directions in a (plane or spherical) control problem, for instance. For large control windows, in flight dynamics for example, this situation is usually aggravated by the fact that opposite sides of a direction rectangle are not parallel.

We further introduce a Cartesian co-ordinate oriented description of directions around the pole as an approximate alternative to a polar-spherical co-ordinate description. If a random direction $\mathbf{v}$ is parameterized by signed arcs on base meridians $\mathcal{M}_j$ at the pole, signed arc lengths will be denoted by ϵ_j and referred to as *surface pseudo co-ordinates* of a direction. The actual projection of ϵ_j on the co-ordinate axis $\mathbf{u}_j$ with $j \neq k$ is

$$x_j \;=\; \sin \epsilon_j, \qquad (1 \leq j \leq k-1) \tag{14}$$

which paradoxically is a direction cosine. Let us first define an arbitrary unit vector $\mathbf{v}_e$ by means of its Cartesian components $(x_{e1}, \ldots, x_{ek})$ where the last component is along the pole by convention. Apart from a simple ambiguity, $\mathbf{v}_e$ is also fully defined by $\epsilon_1, \ldots, \epsilon_{k-1}, \pm\sqrt{1 - \sum_{i=1}^{k-1} x_{ei}^2}$. Due to the fact that $\mathbf{v}_e$ is a unit vector we have $\sum_{i=1}^{k} x_{ei}^2 \;=\; 1$ or equivalently $\sum_{1}^{k-1} x_{ei}^2 \;=\; 1 \;-\; x_{ek}^2$, but $x_{ei} \;=\; \sin \epsilon_i$ for $i \;<\; k$ which makes that

$$\sin^2 \epsilon \;=\; \sum_{i=1}^{k-1} \sin^2 \epsilon_i \tag{15}$$

The co-ordinate properties are only 'pseudo' because the ϵ_i on different base meridians (only orthogonal at the pole and anti pole) are not subject to the Euclidean component composition properties, but to

$$\mathbf{v} \;=\; \sum_{i=1}^{k-1} \sin \epsilon_i \, \mathbf{u}_i \;\pm\; \cos \epsilon \, \mathbf{u}_k \tag{16}$$

and only strictly apply on the base meridians. If we thus have random components uniformly distributed inside given intervals on the different base meridians we have to go back to Cartesian components to construct the actual unit vectors. It may happen though that the $k-1$ random pseudo co-ordinates do not yield a real polar Cartesian component, because the condition

$$\sum_{i=1}^{k-1} \sin^2 \epsilon_i \leq 1 \tag{17}$$

is violated. This will thus have to be prevented by a toss away intervention if the cut off limits allow such a case. It is then important to always perform a complete restart, instead of keeping the co-ordinate(s) which so far satisfy/ies (17) and continue by re-simulating the random number which caused the violation. Omitting this precaution leads to correlation which affects the ultimate uniformity of the directions obtained.

If we want to avoid any toss away, we have to ensure that the largest cut off values combined preclude any violation of (17). To formally represent this 'in advance' constraint, we agree to write $\epsilon_j^- \;<\; \epsilon_j^+$ to represent the cutoff of the pseudo-co-ordinates on the base meridian j or, equivalently $x_j^- \;<\; x_j^+$ on the j-th co-ordinate axis. We get the inequality

$$0 \;<\; 1 \;-\; \sum_{j=1}^{k-1} \sin^2\!\big(\sup(|\epsilon_j^-|,\, |\epsilon_j^+|)\big) \tag{18}$$

which we assume to be not satisfied in the h-meridian algorithm for $k = 4$ hereafter:

h1. $x_1 = \sin\xi_1 = \rho_u^{(1)}$ with $\epsilon_1^- \leq \xi_1 \leq \epsilon_1^+$
h2. $x_2 = \sin\xi_2 = \rho_u^{(2)}$ with $\epsilon_2^- \leq \xi_2 \leq \epsilon_2^+$
h3. if $x_{test} = 1 - x_1^2 - x_2^2 < 0$ go back to step (h1)
h4. $x_3 = \sin\xi_3 = \rho_u^{(3)}$ with $\epsilon_3^- \leq \xi_3 \leq \epsilon_3^+$
h5. if $x_{test} = 1 - x_1^2 - x_2^2 - x_3^2 < 0$ go back to step (h1)
h6. $x_4 = \sqrt{x_{test}}$
h7. reposition the rectangle, if required, using 4 direction cosines to the separate co-ordinate axes of the simulation and apply them to the target (rotated) co-ordinate system.

For S^2, this algorithm produces random points in a concave spherical rectangle whose sides are arcs of small circles. These small circles are perpendicular to the equator and their centers are located on the separate Cartesian co-ordinate axes. The optional constraint (18) suggests that, in practice, one may be better off by locating the rectangle symmetrically around the pole at the price of a repositioning of each random point on the sphere. The direction cosines of these points are their co-ordinates themselves. The repositioning method given in our Note 8 is equivalent to a direct rotation in this case, because the reference axes are mutually orthogonal.

We can, of course, by using the area ratio method generate uniform randomly distributed directions on S^2 in spherical rectangles or polygons whose boundaries are great and/or small circle arcs. The figure should then cover less than a hemisphere, but the small circle arcs may belong to circle centers which have not to be confined to the equator.

CONCLUSIONS

Considering our Note 8 together with this contribution we have extended the definition of pdf's based on polar co-ordinates on the three dimensional sphere with the aim to generate uniformly distributed directions on limited figures bounded by great and small circle arcs. In the simplest case, this was derived from the well known spherical integrals involving pdf's, creating the awareness about both, the nature of the figures which can be treated with very little effort on the one hand and the availability of the means of checking uniformity within narrow bounds on the other hand. In more involved cases, we have presented the simulation of random directions by means of area ratios expressed as a function of variable right ascension which analytically represents the variation of the integration of the cumulative distribution function of a right ascension-pdf's centered around the pole. This novel method was applied to an arbitrary spherical triangle. Using the same technique, we added an algorithm for the generation of random directions inside a small circle segment, which can be adapted to small circle intersections. Apart from the polar co-ordinates we also introduced pseudo surface co-ordinates on the sphere issuing from the pole as origin. This has led to a novel and extremely simple, statistically rigorous algorithm for small circle rectangles whose boundaries are arcs of small circles perpendicular to the equator.

REFERENCE

[1] Mardia K.V., Jupp P.E.,*Directional Statistics*, John Wiley & Sons, New York, 2000.