NOTE 3

Version one posted in the Internet on 22.01.2010

ALTERNATIVE GENERALIZATION OF POLYGONS
and their
GONIOMETRY

Luc Fraiture

Abstract. Known properties of affine Euclidean vector spaces and angular properties of rigid line vector structures are brought together to define a generalization of plane polygons in higher dimensions. The derivation of particular constraints hidden in vector geometry represents the main spin off for particular applied mathematical problems.

1. INTRODUCTION

In this note we introduce a generalization of plane trigonometry in the context of an affine Euclidean vector space over $\mathbb{R}$, whose finite dimension is larger or equal to two. The name 'goniometry' appearing in the title means the mathematical methods involving the handling of angles comprising trigonometry as the special case for the dimensions two and three. The way this generalization is obtained, involves a sum of line vectors which can be arranged so as to build a closed geometrical structure in space. This closed structure is a figure with legs and corners which, in dimension two, coincides with a conventional polygon. In higher dimensions we call it *a hyperpolygon*. The naming *'generalized polygon'* cannot be retained, because it is already in use in graph theory. The hyperpolygons are more basic than polytopes (also called polyhedra) in that they have neither a volume nor a surface. According to the definition we will give for hyperpolygons, they have neither an inside nor a outside and consequently, convexity is meaningless (except in dimension two) for this type of very simple geometrical structures.

We were surprised not to have found the slightest trace of hyperpolygons with whatever denomination in the accessible literature. It is nevertheless unlikely that no mathematician would never have cursorily looked into this geometrical approach. Though, as the note will show, without goniometry the subject is quite basic and not really worth pursuing.

Our motivation is twofold. First, we wish to complete the work started with the generalization of spherical trigonometry by Fraiture and Spindler in a paper with a somewhat misleading title[1]. Second, we notice that the ideas developed in this paper have potential applications in interesting areas of vector geometry and more particularly with respect to estimation as is shown by Fraiture[2]. The argument is as follows. If we have to estimate some of the co-ordinates (called components in an engineering context) of a collection of vectors and we can formulate the estimation problem more or less independently of the absolute orientation and origin of these vectors, then we decrease the number of independent parameters to estimate and improve overall accuracy. This approach and its application is intimately linked to the parameterization of the relevant vectors, which also may, in favorable cases, assimilate some or all potential constraints in an adequate manifold. For this purpose we will study a natural parameterization of hyperpolygons, which consists of side lengths and angles among these sides. We will call these parameters the 'Euclidean parameters' of a hyperpolygon. The analytical properties of this parameterization are in fact the generalization of plane trigonometry.

The note is organized as follows. We first define the very basic notions and derive the simple properties applicable to hyperpolygons. This basis allows us to answer the question about the number of independent parameters required to define an hyperpolygon. In the next step the angular constraints and cosine rule of plane trigonometry are generalized to higher dimensions. Thereafter the generalization of the sine rule involving vector products is derived to verify its usefulness.

2. PRELIMINARIES

This section is devoted to a brief introduction of well known specific features useful when dealing with hyperpolygons described in affine Euclidean vector spaces.

Definitions and Conventions

A n-dimensional Euclidean vector space over $\mathbb{R}$ will be termed *conventional* vector space and denoted by V_n if all vectors refer to a common origin.

A *line or free vector* in an hyperplane H of dimension n is defined by the co-ordinates of its start and end points separately. The co-ordinates of the start and end points correspond to the conventional vectors v_a and $v_b \in V_{n+1}$, respectively. The dimension of the conventional vector space can be larger by one with respect to the dimension of the hyperplane H, because the origin of the conventional vectors does not need to be in H. A line vector can be denoted by $\mathbf{v_{ab}} = (v_a, v_b)$. See for instance in Berger[3], Chapter 2 on affine spaces pp. 33-37.

A *side* (or *leg*) s_i in a hyperplane of dimension n is a finite section of a line (of dimension one) characterized by a line vector (v_{ai}, v_{bi}) whose direction is fixed by the start point v_{ai} and the end point v_{bi}.

The vector $\mathbf{w}_i = v_{bi} - v_{ai}$ is called *algebraic vector* corresponding to the side s_i, see Berger chapter 3 pp. 68-72. The algebraic vectors are signed geometrical entities which differ from conventional vectors in that they do not have a common origin. In this section we will show that they can be treated like common Euclidean vectors. If handled this way, the hyperplane in which they are located is an affine Euclidean vector space of a specified dimension n and will be denoted by A_n, see Berger chapter 8 on Euclidean vector spaces pp. 153-156. Algebraic vectors are explicitly used when introducing basic statics and dynamics and further implicitly employed in vector field applications of mathematical physics. In the present context they will be a short representation of line vectors, allowing a vector treatment.

Two line vectors are said to be concatenated if they have a common point at their start and/or end points.

The points where concatenation occurs are called *corners* or corner points.

Sides are said to be *simply concatenated* if these sides are

- a) *all* in contact with one and only one other leg at each of their two end points.
- b) linked together such that the end point of a previous line vector is connected to the start point of the following vector by convention.

Point (b) has an impact on the interpretation of the angle definition involving vectors in the affine space. Due to the fact that in a conventional vector space all vectors have a common origin, the meaningful angle α_{12} between the vectors v_1 and v_2 is defined at this common origin by the well known Hermitian inner or scalar product:

$$< v_1 \cdot v_2 > = \; ||v_1|| \; ||v_2|| \cos \alpha_{12} \tag{1}$$

with $< v \cdot v >= ||v||^2$. We should be aware that the availability of the cosine function and the angle α_{12} goes beyond the basic axioms of an inner product space, because this is a step into geometry. If an Euclidean vector space is equipped with the cosine function and angles according to (1), we call it a goniometric space[1].

The inner product is one of the essential links between Euclidean vector spaces and goniometry. One of the most important features in this respect is the Gram determinant extensively described by Gantmacher[4]. The Gramian is the corresponding matrix. Let $v_1, \dots, v_m \in V_n$, then *the Gramian* $G(v_1, \dots, v_m)$ is the $(m \times m)$ matrix defined by:

$$G(v_1, \dots, v_m) \;:=\; \begin{bmatrix} < v_1 \cdot v_1 > & \cdots & < v_1 \cdot v_m > \\ \vdots & \ddots & \vdots \\ < v_m \cdot v_1 > & \cdots & < v_m \cdot v_m > \end{bmatrix} \tag{2}$$

The Gramian is a positive semi-definite symmetric matrix. If we express $v_1, \dots, v_m$ in coordinates in any orthonormal basis of V_n and form the $(m \times n)$ matrix $A(v_1, \dots, v_m)$ whose rows are these coordinate representations, then we observe that $G(v_1, \dots, v_m)$ $= A\,A^*$, where the asterisk denotes transposition. Thus, if $m = n$ we have det $G(v_1, \dots, v_n)$ $= \det^2 |v_1, \dots, v_n|$.

Proposition 1. Let $v_1, \dots, v_m \in V_n$, then the Gram determinant has a positive value if and only if $m \leq n$ and rank$\,(v_1, \dots, v_m) = m$ (in other words, when the Gramian is non-singular).

Proof. The proof can be found in many textbooks and also in reference 1. ■

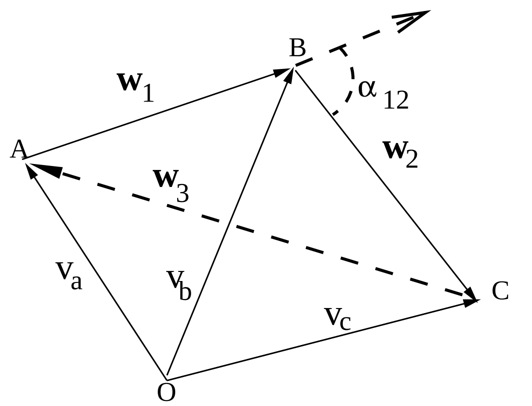

Figure 1. Referencing the affine vectors w_1 and w_2

We would now like to know whether we can use the angle definition (1) and the Gramian for line vectors in any finite dimension. Therefore we consider Fig. 1. Reflecting on the geometrical significance of (1) we see that it is essential that the vectors have a common origin and thereby enclose the angle α_{12} at that origin. Consider now the concatenated line vectors $\mathbf{w_1}$ and $\mathbf{w_2}$, while v_a, v_b, v_c are conventional vectors with the common origin in O. For the only common point between $\mathbf{w_1}$ and $\mathbf{w_2}$ is B, the only meaningful angle α_{12} between these line vectors is actually formed by the shifted vector $\mathbf{w_1}$ with $\mathbf{w_2}$. Due to the necessity to work with a directed concatenation means that, in contrast to the convention in common basic Euclidean geometry of polygons, the angles at the corners of hyperpolygons are the supplement of the conventional angles, as shown in Fig. 1. But inner products of non-concatenated affine vectors can mathematically only be made in terms of algebraic vectors, and to understand their geometrical and goniometric significance we need the following definitions.

A *translation* $D(x)$ of a line vector $\mathbf{v_{ab}}$ in the hyperplane H of dimension n by a vector $x \in H$ is the map $D(x)\mathbf{v_{ab}} := (v_a + x, v_b + x)$.

The line vector $\mathbf{v_1} = (v_{a1}, v_{b1})$ is said to be *collinear (parallel)* with the line vector $\mathbf{v_2} = (v_{a2}, v_{b2})$ if and only if $< (v_{b1} - v_{a1}) \cdot (v_{b2} - v_{a2}) > /(||v_{b1} - v_{a1}||\ ||v_{b2} - v_{a2}||) = \pm 1$.

Proposition 2.

a) An arbitrary translation of a line vector leaves the corresponding algebraic vector unchanged.

b) The origin (start point) of two line vectors can be brought together by means of a translation of one of the line vectors.

Proof. The proof of (a) is trivial. For (b) we start from the line vectors $\mathbf{v_1} = (v_{a1}, v_{b1})$ and $\mathbf{v_2} = (v_{a2}, v_{b2})$. We now apply a translation $D(v_{a1} - v_{b2})$ to $\mathbf{v_2}$ and the origin of both line vectors becomes equal. Thereby they become equal to two conventional vectors as claimed. ■

Corollary. Algebraic vectors remain the same whether the corresponding line vectors have a common origin or not and the algebraic vectors satisfy the axioms of conventional vectors. Moreover, these claims show that algebraic vectors satisfy the axioms of conventional vector spaces. In the present context, it is nevertheless indicated to maintain the the name 'algebraic vector' for vectors belonging to hyperpolygons, but bold faces, as proposed before, can be omitted.

3. HYPERPOLYGONS

We start this section by giving a simple example. It is not difficult to imagine the construction of a figure having one more leg than a triangle in dimension $n = 3$. This

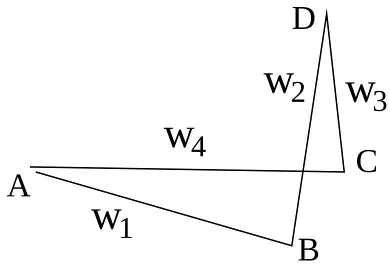

Fig. 2 Example of a three dimensional quadrangle.

construction is still representable in our world. Consider an arbitrary triangle. Remove one side and replace it by two sides connected to a point added outside the plane of the triangle. The result is a quadrangle depicted in figure 2 where D is not inside the plane defined by A, B and C. The sides of ABCD taken together actually form what we will define later to be a *polygonal simplex* in dimension three.
Definition

It is further a consequence of the definition of simple concatenation, that a collection of legs, which are *all* mutually simply concatenated, *necessarily* forms a closed structure, which we will call *hyperpolygon*.

An ordered collection of $m \geq 3$ conventional non-trivial vectors $(v_1, \ldots, v_m) \in V_n$, where no two subsequent vectors v_i, v_{i+1} for $1 \leq i \leq m$ with $v_{m+1} = v_1$ are allowed to be collinear, and which is subject to $2 \leq \text{rank } (v_1, \ldots, v_m) \leq m \leq n$ defines an hyperpolygon with the algebraic vectors $(w_1, \ldots, w_m)$ by means of the following transformation:

$$\begin{aligned} w_1 &= v_2 - v_1 \\ w_2 &= v_3 - v_2 \\ &\ldots \\ w_{m-1} &= v_m - v_{m-1} \\ w_m &= v_1 - v_m \end{aligned} \tag{3}$$

$$\sum_{i=1}^{m} w_i = 0 \tag{4}$$

We say that $(v_1, \ldots, v_m)$ is a collection of *generating vectors of the hyperpolygon* represented by (4), provided no v_i is collinear with one of its immediate neighbors and consequently that no resulting algebraic vector is identically zero.
If it happens that the generating vectors add up to zero it is pointless to apply the transformation (3), because they can be considered to be the algebraic vectors of a hyperpolygon.
Proposition 3. It is not possible that one algebraic vector is orthogonal to all the other algebraic vectors of one and the same hyperpolygon.
Proof. Should this not the case, then multiplying (4) by such an orthogonal vector, say w_1, yields $0 =< w_1 \cdot w_1 >$, which implies $w_1 = 0$. This is excluded by definition. ■

To introduce the basic facts about the rank of the generating vectors and the corresponding algebraic vectors we reconsider Fig. 1 where (v_a, v_b, v_c) can be considered to be the generating vectors of the triangle ABC. The sides of this triangle are the algebraic vectors denoted by (w_1, w_2, w_3). The rank of (w_1, w_2, w_3) is two but the rank of the generating vectors (v_a, v_b, v_c) is either two if OABC are laying in a two dimensional plane, or has rank three if C does not belong to the plane of AOB. These typical differences are special cases of the following properties.
Proposition 4. Consider an ordered collection of generating vectors $(v_1, \ldots, v_m) \in V_n$ with rank $(v_1, \ldots, v_m) = M_R \leq n$ and the corresponding algebraic vectors $(w_1, \ldots, w_m)$ with rank $(w_1, \ldots, w_m) = W_R$, then
a. we have necessarily $W_R = M_R - 1$ if $m = M_R$ and $m \leq n$,
b. in order that $W_R = n$, it is necessary but not sufficient to have $n + 1 \leq m$

Proof. In the case of (a) we first remark that due to (4) $W_R < m$ if $M_R = m$. But if on top of $(w_1, \ldots, w_m)$ we just know one more vector, take for instance v_1, we are able to reconstruct $(v_1, \ldots, v_m)$ from $(w_1, \ldots, w_m, v_1)$ just using the equations (2), or rank $(w_1, \ldots, w_m, v_1) = m$ and hence claim (a) is verified. The validity of claim (b) follows from (a). ■

Definitions

A m-sided hyperpolygon with the ordered sides $s_1, ..., s_m$ in A_n is said to be *embedded* if the algebraic vectors corresponding to these sides do not span the vector space A_n, or $\mathrm{rank}(w_1, ..., w_m) < n$. Otherwise we say that the hyperpolygon *has full rank.* In principle, the vectorial description of an embedded hyperpolygon can isomorphically be mapped into a full rank structure by means of an orthogonal transformation.

We define the generalization of triangles to be a hyperpolygon with m sides with rank $W_R = m - 1$. We already agreed before to call them *polygonal simplices.* In a vector space A_n a full rank polygonal simplex has $n + 1$ sides according to proposition 4.

Proposition 5. If $\mathcal{P} = (w_1, \ldots, w_m)$ is a polygonal simplex, then any collection $Q \subset \mathcal{P}$ of $p \leq m - 1$ algebraic vectors has rank $(Q) = p$.

Proof. We first turn our attention to the case $m - 1 = p$. Assume first that the collection $Q_a = (w_1, \ldots, w_{m-1})$ has $\mathrm{rank}(Q_a) = m-1$, while $Q_b = (w_2, \ldots, w_m)$ has only $\mathrm{rank}(Q_b) = m_b = m - 2$. But then v_m can be expressed as a linear combination of the elements of $Q_c = (v_2, ..., v_{m-2})$. Let $(e_2, \ldots, e_{m-1})$ be the corresponding basis for Q_c. Hence we can decompose the two vectors v_1 and v_m as follows:

$$v_1 = \sum_{i=2}^{m-1} a_i \; e_i \; + \; v_o$$

$$v_m = \sum_{i=2}^{m-2} b_i \; e_i$$

Substituting this into (4) yields:

$$v_o = -\sum_{i=2}^{m-1} [(a_i \; + \; b_i)e_i + v_i]$$

But $v_o \notin span(Q_c)$ while all algebraic vectors on the right hand side of this equation only belong to $span(Q_c)$. This is only possible if both members are identically zero, which is not possible. The argument can be repeated for any combination, proving the claim for $m_b = m - 1$. Now we can start from a collection of m linearly independent vectors out of which $p < m$ vectors are selected. By the definition of linear independence these p vectors must have rank p, completing the proof. ■

Corollary. Exactly as for a triangle, polygonal simplices cannot have any collinear sides. This does, of course, not apply to a m-sided hyperpolygon whose rank is smaller than $m - 1$.

4. PARAMETERIZATIONS

In this section we will determine the number of independent scalar parameters we need to specify a collection of generating vectors on one hand and the corresponding collection of algebraic vectors of the corresponding hyperpolygon on the other hand. This is motivated by applied mathematical problems where we have some n vectors which require a reference system to be able to work with them. Now this reference system could be replaced partly or completely by exploiting some natural properties hidden in the vectors considered. The introduction of hyperpolygons is one way which can remove some features of an exterior reference and thereby lead to a reduction of descriptive parameters. To introduce this subject, we have quickly to recall a few well known basics applicable to line vector figures in affine vector spaces.
Definitions.

Two m-sided hyperpolygons $\mathcal{P}_a$ and $\mathcal{P}_b$ in the n-dimensional hyperplane with the corner point co-ordinates $(v_{a1}, \ldots, v_{am})$ and $(v_{b1}, \ldots, v_{bm})$ respectively, are said to be *congruent* if there exists a translation D and a map $T \in O^+(n)$, such that $D(Tv_{bi}, Tv_{bi+1}) = (v_{ai}, v_{ai+1})$ for all $1 \leq i \leq m-1$ (this type of transformation is commonly known as an *affine transformation*). The two m-sided hyperpolygons $\mathcal{P}_a$ and $\mathcal{P}_b$ are *mirror* congruent if $T \in O^+(n)$ is replaced by $T \in O^-(n)$ in the previous definition. (Congruence corresponds to isometry if we compare two sets of conventional vectors in V_n).

Two m-sided hyperpolygons $\mathcal{P}_a$ and $\mathcal{P}_b$ in the n-dimensional hyperplane H with the corner point co-ordinates $(v_{a1}, \ldots, v_{am})$ and $(v_{b1}, \ldots, v_{bm})$, respectively, are said to be *(mirror)-similar* if there exist a non-zero scaling factor $\lambda \in \mathbb{R}$ such that $(v_{a1}, \ldots, v_{am})$ is (mirror)-congruent with $(\lambda v_{b1}, \ldots, \lambda v_{bm})$. (Similarity corresponds to homomorphism if we compare two sets of conventional vectors in V_n).

Proposition 6.
(a) If the set $\mathcal{Q} = (v_1, \ldots, v_n) \in V_n$ has rank n, then this set can be represented unambiguously up to isometry by $0.5n(n+1)$ scalar parameters.
(b) for any k supplementary vectors added to $\mathcal{Q}$ we need kn supplementary independent scalar parameters to characterize the augmented set $\mathcal{Q}$ completely.

Proof. (a) We can put the vector v_i into the i-th column of the $n \times n$ matrix A and for rank $\mathcal{Q} = n$ also rank $(A) = n$. It is well known that in this case there exists an isometric map $T \in O(n)$ such that $B = TA$ is lower triangular (obtained for instance by means of a sequence of Householder transformations). The column vectors of A and B are congruent/isometric by construction. Consequently, to characterize B we only need to know the value of the $0.5n(n+1)$ components of the lower triangle, thus requiring the number of parameters as claimed. (b) The n vectors which are in the columns of B can be used as a complete basis of V_n and every new vector added requires kn co-ordinates (components) in this basis to be completely specified, as claimed. ■

Corollary. We note that $0.5n(n+1) = n^2 - 0.5n(n-1)$. This means that characterization in the context of proposition 5 requires all vector co-ordinates minus a number corresponding to the number of scalar parameters defining an orthogonal transformation in V_n.

THEOREM 1
The characterization up to similarity of a m-sided hyperpolygon in A_n with $n \leq m-1$

requires

$$N(n)_m \;=\; nm - 2n - 0.5(n-1)(n-2) \tag{5}$$

mutually independent scalars.

Proof. This theorem relies on the fact that a hyperpolygon is by definition constrained by (4). We thus start from the m algebraic vectors w_i. Their specification requires nm scalar parameters. But (4) acts as constraint supplying n homogeneous scalar equations. In these equations we can further select one vector to take any arbitrary but non zero values. This is legitimate, provided the adequate rotation is applied to the other sides as well (congruence) and similarly the scale factor leading to the norm of this one vector has to be applied to the other sides as well (similarity). These arbitrary co-ordinates transform (4) into n inhomogeneous equations. These equations and the selected vector constrain $2n$ parameters. Finally according to proposition 6, the whole hyperpolygon can be rotated around the vector chosen freely, allowing for an arbitrary isometric map $T \in O^+(n-1)$ defined by $(n-1)(n-2)/2$ scalar parameters. Subtracting the number of constrained and arbitrarily selectable parameters from nm leaves (5). ■

By inspecting (5) we note that

$$2n \,+\, 0.5\,(n \,-\, 1)\,(n \,-\, 2) \,=\, n \,+\, 0.5\,n\,(n \,-\, 1) \,+\, 1$$

This means that the decrease of the number of actually independent parameters with respect to nm in (5) correspond to a translation, namely n, an orthogonal transformation, namely $0.5\,n\,(n-1)$ and finally a scaling factor. Apart from the scaling factor, this is the attitude of a rigid body described by a translation and rotation as was already analytically derived by Euler[5,6] for dimension three (usually attributed to Chasles, who gave a geometrical prive).

By considering the m generating vectors on their own, instead of the resulting hyperpolygon, we also start from nm parameters. In favorable cases it is possible to reformulate the problem studied, such that we can hide an orthogonal transformation $O^+(n)$ and save $0.5n(n+1)$ parameters in the resolution phase. This can be achieved if particular vectors, playing a role in a given problem, can be used as basis vectors[7]. The hyperpolygon then still has the supplementary advantage of the free translation and scaling or, in other words, needing $n+1$ parameters less. This extra gain is due to the fact that we need one arbitrary generating vector and an adequate scaling factor to perform the inverse transformation from the hyperpolygon back to the adequate generating vectors.

Definition

All the relative (or dimensionless) side lengths s_i/s_1 and all the cosines $\cos\alpha_{ij}$ of a m-sided hyperpolygon whether embedded or not are called *the Euclidean parameters* of the hyperpolygon.

Proposition 7.

a) The total number of Euclidean parameters of a m-sided hyperpolygon in A_n with $n \leq m-1$ amounts to

$$P_m = 0.5(m+2)(m-1)$$

b) When the number of all Euclidean parameters belonging to $(m{+}1)$-sided hyperpolygon is

compared with a m-sided hyperpolygon, both with the same rank $= n$, the former requires $m+1$ more Euclidean parameters, or $P_{m+1} = P_m + m + 1$ against only n supplementary algebraically independent parameters, or $N(n)_{m+1} = N(n)_m + n$.
Proof. For claim (a) we note that the number of relative side lengths is $m-1$ while the total number of angles is $0.5m(m-1)$. They add to $P_m = 0.5(m+2)(m-1)$ as proposed. Claim (b) results trivially from (a) and the previous theorem. ■

Corollary. We have

$$2(P_m - N_m) \;=\; (m-n)^2 + m + n$$

or by writing $m = n + \mu$

$$P_m - N(n)_m \;=\; 0.5\mu(\mu+1) + n$$

For a polygonal simplex this means:

$$P_{n+1} - N(n)_{n+1} \;=\; n+1$$

Altogether, this means that we have plenty of Euclidean parameters. Consequently, they must be dependent to the extent required by theorem 1. These dependencies must result in constraints. If such constraints are tractable, they may be used in applied mathematical problems starting from at least three arbitrary vectors employed as generating vectors to derive a (hyper)polygon. In this respect we stress that especially hyperpolygons in dimension three may lead to useful insights.

The analytical expression for these constraints can be gained by exploiting three sources. The first of these sources is the Gram determinant which leads to the generation of constraint expressions involving only angles. This will be dealt with in the next section. The second type of constraints results from the vector equation (4) yielding generalizations of the cosine rule. This will be handled in section 6. Finally, we can also make use of the vector product providing the generalized sine rules.

5. THE ANGULAR CONSTRAINT RULES

Consider a m-sided hyperpolygon characterized by the algebraic vectors $(w_1, \ldots, w_m)$ and let us agree that $w_i = \ell_i\, \hat{w}_i$, where ℓ_i is the positive length of the side s_i or equivalently $\hat{w}_i$ is a unit vector. Hence, the Gramian can be decomposed as follows:

$$G(w_1, \ldots, w_m) \;=\; U(\ell_1, \ldots, \ell_m)\; G(\hat{w}_1, \ldots, \hat{w}_m)\; U(\ell_1, \ldots, \ell_m) \tag{6}$$

where $U(\ell_1, \ldots, \ell_m)$ is the $m \times m$ diagonal matrix with the non-zero elements $u_{ii} = \ell_i$ and

$$\hat{G}_m \;=\; G(\hat{w}_1, \ldots, \hat{w}_m) \;=\; \begin{vmatrix} 1 & \cos\alpha_{12} & \cdots & \cos\alpha_{1m} \\ \cos\alpha_{12} & 1 & \cdots & \cos\alpha_{2m} \\ \vdots & & \ddots & \vdots \\ \cos\alpha_{1m} & \cos\alpha_{2m} & \cdots & 1 \end{vmatrix} \tag{7}$$

employing the fact that $\alpha_{ij} = \alpha_{ji}$ in real goniometric spaces. We will call $\hat{G}_m$ a goniometric Gramian. From (6) and (7) it follows that

$$\operatorname{rank} \hat{G}_m \;=\; \operatorname{rank} G(w_1, \ldots, w_m) \tag{8}$$

which leads to the following important theorem.

THEOREM 2.

The complete set of angles of any valid m-legged hyperpolygons of rank n with $0 < \mu = m - n$ is subject to one or more constraints independently of the length of the legs.

Proof. If $\hat{G}_m$ has only rank n, and $n < m$, this means that the determinant of all minors of $\hat{G}_m$ of dimension $(n+p) \times (n+p)$, with $1 \leq p \leq \mu$, are equal to zero. Each of these zero determinants is equivalent to an angular constraint equation. ■

Corollary a. While (7) and (8) also applies to non-trivial conventional vectors and the application of (4) is not implied in the proof, theorem 2 also applies to sets of conventional vectors provided these sets are rank deficient.

Corollary b. The totality of the angles of a polygonal simplex in A_n is subject to the only condition: $\det \hat{G}_{n+1} = 0$.

Corollary c. if we know all angles except one for a full rank hyperpolygon, the missing angle can be derived from $\det \hat{G}_m = 0$ with a simple ambiguity.

Corollary (b) can easily be verified in the case of a triangle, where

$$\det \hat{G}_3 = 1 + 2\, \cos\alpha_{12}\, \cos\alpha_{13} \cos\alpha_{23} - \cos^2\alpha_{12} - \cos^2\alpha_{13} - \cos^2\alpha_{23}$$

and by setting $\alpha_{13} = 2\pi - \alpha_{12} - \alpha_{13}$ the previous equation becomes

$$1 + \cos(\alpha_{12} + \alpha_{13})\, \cos\alpha_{12}\, \cos\alpha_{13} - \cos^2(\alpha_{12} + \alpha_{13}) - \cos^2\alpha_{13} - \cos^2\alpha_{23} \equiv 0$$

confirming the known condition $2\pi = \alpha_{12} + \alpha_{13} + \alpha_{13}$.

4. THE COSINE RULES

To derive the cosine rules we start from (4) which we left multiply with w_k to obtain

$$\ell_k^2 \ + \sum_{i \neq k}^{m} \ell_k \ell_i \ \cos\alpha_{ki} \ = \ 0$$

or simply

$$\ell_k \ + \sum_{i \neq k}^{m} \ell_i \ \cos\alpha_{ki} \ = \ 0 \qquad\qquad (1 \leq k \leq m) \tag{9}$$

yielding $m \geq n+1$ bilinear equations in the variables ℓ_i and $\cos\alpha_{ij}$. We propose to call these equations *"the projection rules"*. To be in line with the convention made for Euclidean parameters we have to specify the length of an arbitrary side. We chose $\ell_1 = 1$ in all what follows, even if we write ℓ_1 explicitly.

A second cosine rule is obtained by splitting (4) in two members, namely $\sum_{j=1}^{p} \ w_{aj} \ = -\sum_{i=1}^{n-p+1} \ w_{bi}$ with $w_{aj} \neq w_{bi}$ for any i and j. Upon squaring both members we obtain:

$$\sum_{j=1}^{p} \ell_{aj}^2 + 2\sum_{r \neq q} \ell_{ar}\ell_{aq} \cos\alpha_{ar,aq} = \sum_{i=1}^{n-p+1} \ell_{bi}^2 + 2\sum_{t \neq s} \ell_{bt}\ell_{bs} \cos\alpha_{bt,bs} \tag{10}$$

We propose to call these equations "*the generalized cosine rules*". Including the case $p = 0$ there are obviously 2^n different quadratic cosine rule equations. At a first glance we could believe that the quadratic cosine rules do not really add anything to the projection rules which are bilinear, while the latter are quadratic. Closer inspection shows however, that the quadratic cosine rules offer alternative parameter resolutions which may be useful in practice.

6. INTERRELATIONS BETWEEN ANGLES AND SIDE LENGTHS

In this section we consider the problem of completing our knowledge of a hyperpolygon starting from very particular samples of Euclidean parameters and making use of the the analytical constraints derived so far. Let us start with the cases where we know the angles and want to find the relative side lengths assuming $\ell_1 = 1$.

Proposition 8. If the $0.5n(n+1)$ values of $\cos\alpha_{ij}$ of a full rank m-sided hyperpolygon in A_n are known, then all side lengths can be determined uniquely provided $m - n = \mu$ relative side lengths, including $\ell_1 = 1$, are known and, in the case $1 < \mu$, that there is at least one selection of n sides which has rank n without being all mutually orthogonal.

Proof. Let us first consider the case of a polygonal simplex. This corresponds to $\mu = 1$. Then $\operatorname{rank}\hat{G}_m = \operatorname{rank}\hat{G}_{n+1} = n$. This means that we can write

$$0 = \hat{G}_{n+1}\,|\ell_1, \ldots, \ell_{n+1}|' \tag{11}$$

But due to proposition 5, any n algebraic vectors of a full rank polygonal simplex have rank n, which means that any $n \times n$ minor of $\hat{G}_{n+1}$ is non-singular. Consequently, we are free to choose the equations from the second row of (11) onwards and rearrange them as follows

$$\begin{vmatrix} 1 & \cos\alpha_{23} & \cdots & \cos\alpha_{2m} \\ \cos\alpha_{32} & 1 & \cdots & \cos\alpha_{3m} \\ \vdots & & \ddots & \vdots \\ \cos\alpha_{m2} & \cos\alpha_{m3} & \cdots & 1 \end{vmatrix} \begin{vmatrix} \ell_2 \\ \vdots \\ \ell_{n+1} \end{vmatrix} = - \begin{vmatrix} \cos\alpha_{12} \\ \vdots \\ \cos\alpha_{m1} \end{vmatrix} \tag{12}$$

where the matrix of the left hand side of (12) has full rank. But in order to get a unique determination of the relative lengths, the vector on the right hand side of (12) needs to be non-zero. This is secured by the property that no algebraic vectors of a polygonal simplex can be orthogonal to another vector of the same hyperpolygon and thus none of the different $\cos\alpha_{ij}$ is equal to zero. This proves the proposition for a full rank polygonal simplex. ■

Consider now the case $1 < \mu$. The goniometric Gramian has still rank n but its size has increased and we can no longer claim that n arbitrary rows of $\hat{G}_m$ have rank n. But there is at least one such selection, otherwise the hyperpolygon is not full rank. Thus a splitting of the goniometric Gramian, as in (12), is feasible, but involving now the μ columns corresponding to the μ known side lengths. The right hand side of the resulting equation system organized like in (12) is not equal to zero for at least one selection of n sides, because otherwise all sides of the chosen selection must be orthogonal, what we have excluded. ■

The hyperrectangles, which we shall shortly address in the section 'concluding examples', do never satisfy proposition 7 in contrast to rectangular polygonal simplices having only one (hyper)-hypothenuse, which always fulfill the conditions of the proposition just made.

When looking at the hyperpolygon starting from 'all side lengths known' we first observe that if all side lengths of a triangle are known only two mirror congruent figures are possible. In fact, a triangle is a stiff figure. By this we mean that it cannot be deformed if we assume the sides to be undeformable rods of zero thickness and fitted together at their ends by hinges allowing unconstrained rotations in any direction. This rigidity of the triangle has to do with the fact that the internal angles at the corners add up to π or, equivalently, the external or goniometric angles of the corresponding line and algebraic vectors add up to 2π. Although the guidance provided by the properties of the triangle has hitherto been trustworthy to unveil the properties of polygonal simplices in higher dimensions, this is not the case here. To show this we reconsider Fig. 2 in dimension three. We immediately notice that we could fold the triangle BCD in the direction of the plane of triangle ABC by means of a rotation around the fictitious axis BC. This type of folding is also possible in higher dimensions, because BCD could be a part of a higher dimensional polygon and by folding the BDC triangle around the fictitious axis BC we modify the angles of the hyperpolygon without changing the length of the sides. The only conclusion of interest in the present context claims that polygonal simplices from dimension three onwards and 'a fortiori' any other hyperpolygon are not stiff from dimension two onwards, and thus they are rather undefined by the mere knowledge of all side lengths.

The argument used to show the lack of stiffness of most hyperpolygons allows us to make a little side step in the area of folding which is in fact an inelastic deformation. By folding together we mean a sequence of foldings ending in a figure in dimension two or one. The simplest case is the rotation of the triangle BCD onto the plane of triangle ABC in Fig. 1. Thereby we immediately end in dimension two. In an higher dimension the BDC triangle could also be rotated around the fictitious axis BC onto the plane containing BC and the side concatenated at the corner C. Also this can achieve a reduction by one of dimension of the polygon concerned. This reduction is certain for a polygonal simplex as one easily verifies. This kind of folding can be continued up to dimension two, where rotation axes collapse to rotation centers or points. Our geometrical phantasy may even suggest to go a step further into an excursion from hyperpolygon geometry to lattice structures including dimensions higher than three. We do not want to pursue these thoughts, which go far beyond the scope of the present note and leave this subject to the fortuitous interest these few sentences may have generated.

In the event that we know some or all side lengths and some angles, the projection rules in (9) can be reorganized so as to set up an equation system where certain unknowns can be determined, because, for a full rank hyperpolygone, n equations of the projection rules are linearly independent. It is a matter of the particular application one is facing, whether a feasible solution can be found to the equally particular unknowns one wants to determine. In dimension three the verification of this question is tractable, while problems in higher dimensions rapidly get too intricate, except if the hyperpolygons happen to be rank deficient and can be projected into dimension three or even into dimension two.

6. THE SINERULES

Just like the sine rule applies to (plane) triangles in dimension two, the generalized sine rule applies to polygonal simplices and happens to be quite simple when employing vector products. We thereby leave the constraints derived in terms of Euclidean parameters and turn to vectors, both conventional and algebraic. Generalities about the vector product in higher dimensions are available in many textbooks[3], but here we would prefer to employ an approach based on a definition involving a classical determinant function[1]. This function maps a collection of n vectors $(u_1, \ldots, u_n) \in V_n$ onto a scalar (the determinant value) which is not identically zero and alternating such that : $\det(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = \; sign(\sigma)\det(u_1, \ldots, u_n)$, for all elements $\sigma \in Sym_n$ of the symmetric group on n elements and for all $u_i \in V_n$. For $n-1$ vectors $u_1, \ldots, u_{n-1} \in V_n$ the vector product denoted by $u_1 \times \ldots \times u_{n-1}$ is defined by

$$< d \cdot (u_1 \times \ldots \times u_{n-1}) > = \; \det(d, u_1, \ldots, u_{n-1}) \tag{13}$$

for any $d \in V_n$. This may sound quite difficult, but the case of dimension three may elucidate the situation. Consider the collection of vectors $(a, b, c) \in V_3$ where c is equal to the vector product of a and b, then the previous definition says

$$||c||^2 \; = \; ||a \times b||^2 \; = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ (a \times b)_x & (a \times b)_y & (a \times b)_z \end{vmatrix} \tag{14}$$

which obviously means that the component c_j of the vector product is in this case equal to $(-1)^{3+j}\, M_{3j}$, where the minor M_{ij} is the new determinant obtained by deleting row i and column j of the original determinant.

For the properties of vector products in dimension three are well known in applied mathematics, we believe that it is sufficient to recall the most basic properties of a vector product in any finite dimensional vector space V_n, namely:

a - the vector product $u_1 \times \ldots \times u_{n-1}$ is orthogonal to any of the vectors u_i with $i = 1, \ldots, n-1$,

b - the vector product is zero if and only if $u_1, \ldots, u_{n-1}$ are linearly dependent,

c - if T is an isometric map or equivalently an orthogonal transformation in $\mathbb{R}$, then $T(u_1 \times \ldots \times u_{n-1}) = Tu_1 \times \ldots \times Tu_{n-1}$.

d - due to the alternating nature of the sign of a determinant in function of the ordering of its rows, cyclic permutations inside a mixed product remain equal in absolute value and more precisely:

$$< d \cdot (u_1 \times \ldots \times u_{n-1}) >= \; (-1)^{n+1} < u_{n-1} \cdot (d \times u_1 \times \ldots \times u_{n-2}) > \tag{15}$$

quoted as example for u_i and $d \in V_n$

We start by describing the well known sine rule in dimension two, or $n = 2$, making use of the vector product. Therefore we consider the non-zero arbitrary vector $a(a_x, a_y)$. According to the definition given before, the vector product involves $n-1$ vectors. Here this corresponds to just one vector for which we take a. Assume that the vector product is equal to the vector d. Rewriting (14) for this case, yields:

$$||d||^2 \;=\; ||a \times \cdot||^2 \;=\; \begin{vmatrix} a_x & a_y \\ d_x & d_y \end{vmatrix}$$

This means that

$$d_x \;=\; (-1)^{2+1} a_y \;=\; -a_y \qquad d_y \;=\; (-1)^{2+2} a_x \;=\; a_x$$

or $d \perp a$ and $||a|| = ||d||$. Let a be an algebraic vector for the representation of a triangle involving the sides a, b, c with $a + b + c = 0$. Then, by inspecting Fig. 3,

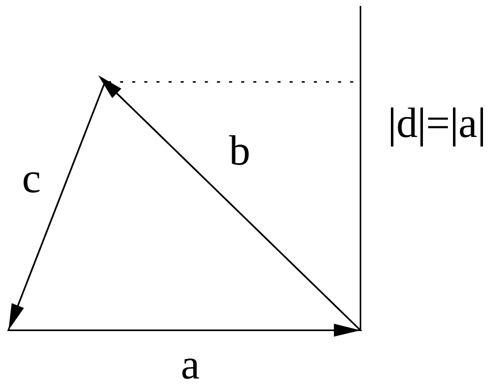

Fig. 3. the sine rule vector geometry in dimension two

we see that $< b \cdot d >= - < c \cdot d >$ must geometrically hold and this corresponds to the essence of the sine rule. We intentionally avoid to write the sine rule explicitly, because the sine loses its geometrical sense once we leave trigonometry[1] and it then merely becomes a short hand for $\pm\sqrt{1 \,-\, cosine^2}$.

What occurs in dimension two is exactly what happens as well for any polygonal simplex in A_n, because a vector product among sides of the polygonal simplex involves $n-1$ sides and the polygonal simplex has two more sides necessarily not involved in the vector product. Thus, assume such a figure consisting of the vectors $(w_1, \ldots, w_{n+1})$, and, for example, a vector product among the first $n-1$ sides, then:

$$0 \;=< (\sum_{i=1}^{n+1} w_i) \cdot (w_1 \times \ldots \times w_{n-1}) > \tag{16}$$

Due to (4) and property (a) of the vector products, this is equivalent to

$$< w_n \cdot (w_1 \times \ldots \times w_{n-1}) >= \,-\, < w_{n+1} \cdot (w_1 \times \ldots \times w_{n-1}) > \tag{17}$$

This is what we meant to be one of the many appearances of the sine rules for a polygonal simplex in dimension n. Apart from a potential sign adaptation each of the equation members can separately be modified by up to $n!$ permutations (automatically including cyclic permutation) according to property d given before. To find the number of independent sine rules we observe that w_{n+1} is missing in the left hand member of (17) for whatever permutation in that member, while w_n fails on the right hand side. We may assume that a similar equation, where w_{n-1} is missing on the left and w_n on the right, is independent from (17), because no equality can be reached by means of permutations. Continuing this sequence in the same way leads to n independent sine rules altogether. Applied to Fig. 2, we obtain

$$\begin{aligned} < w_1 \cdot (w_3 \times w_4) > &= - < w_2 \cdot (w_3 \times w_4) > \\ < w_2 \cdot (w_4 \times w_1) > &= - < w_3 \cdot (w_4 \times w_1) > \\ < w_3 \cdot (w_1 \times w_2) > &= - < w_4 \cdot (w_1 \times w_2) > \end{aligned} \tag{18}$$

This may seem a nice and useful result, but in reality it is disenchanting. Based on (4) we can, for instance, substitute $-w_1 - w_3 - w_4$ for w_2 in the first equation and this leads to an identity. A similar procedure gives a similar result for the other two equations. This means that the construction of the algebraic vectors on the basis of four generating vectors, for a four dimensional polygonal simplex, implies only three independent algebraic vectors, whose negative sum can be substituted for the fourth in all relevant equations and inner products. In principle, (17) is indirectly doing the same but in an intricate manner. In practice we should thus dispense with the sinerule generalization for full rank polygonal simplices in V_n and replace it by a simple elimination of one algebraic vector, realizing a decrease of independent parameters by n. The situation is different if the full rank hyperpolygon is not a polygonal simplex, or $n + 1 < m$, because then such a simple substitution is no longer applicable. In such cases not any sample of $n - 1$ algebraic vectors has rank $n-1$, but for each collection which has rank $n-1$, (16) applies and yields equations involving usually more than two meaningful mixed products. This is, of course, already true in dimension three.

CONCLUDING EXAMPLES

In this last section we wish to give a couple simple examples of hyperpolygons in dimension three, the higher dimension which is closest to practice.

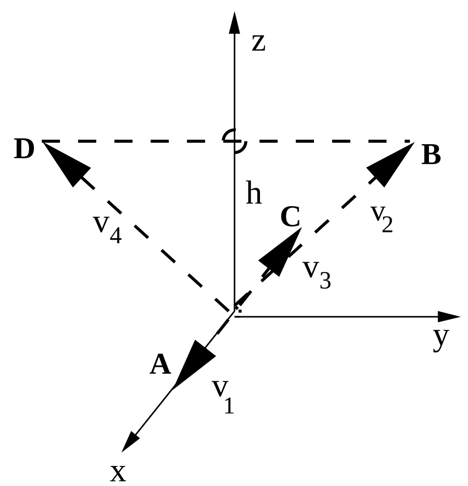

Fig. 4 Generating vectors of a three dimensional polygonal simplex

To start with we consider a partly symmetric polygonal simplex starting with its four generating vectors v_1 with the co-ordinates at the corner A in Fig. 4 corresponding to v_1, v_2 at the corner B, v_3 at C and v_4 at D, namely:

$$\begin{aligned} v_1' &= |\ \ 0.5, 0.0, 0.0|, & v_2' &= |0.0,\ \ 0.5, h| \\ v_3' &= |-0.5, 0.0, 0.0|, & v_4' &= |0.0, -0.5, h| \end{aligned}$$

where AC and DB have length one and the height h of of the line segment DB will be fixed later. By applying the transformation (3) we obtain the algebraic vectors for the sides of the polygonal simplex. These are:

$$\begin{aligned} w_1' = v_2' - v_1' &= |-0.5, 0.5, h|, & w_2' = v_3' - v_2' &= |-0.5, -0.5, -h| \\ w_3' = v_4' - v_3' &= |\ 0.5, -0.5, h|, & w_4' = v_1' - v_4' &= |\ 0.5,\ 0.5, -h| \end{aligned}$$

By closely inspecting the components contained in the algebraic vectors, we see that in the case of w_1, for instance, they are simply the co-ordinates of the end point of the line segment AB which is shifted parallel to itself so that A coincides with the origin.

If we now want a regular or fully symmetric polygonal simplex, this means a figure corresponding to the concatenated sides of a regular simplex minus the sides AC and DB, we need that all sides of the polygon have length one, to be consistent with our choice AC=DB=1. This yields the condition $2(0.5)^2 + h^2 = 1$, or $h = \sqrt{0.5} = \cos 45^0$. We derive the angles at the concatenation points by computing $< w_i \cdot w_{i+1} >= -0.5 = \cos 120^0$ where in the case of $i = 4$ we set $i + 1$ equal to one. The angle between the not adjacent sides is easily found from $< w_1 \cdot w_3 >=< w_2 \cdot w_4 >= -0.5 + (\cos 45^0)^2 = 0$ which means that these sides are mutually perpendicular in A_3 according to our definitions. Knowing the value of $< w_i \cdot w_{i+1} >$, the value x of $< w_1 \cdot w_3 >=< w_2 \cdot w_4 >$ can also be found from the corresponding goniometric Gramian $\hat{G}_4$, because

$$\det(\hat{G}_4) \;=\; 0 \;=\; \det \begin{vmatrix} 1 & -0.5 & x & -0.5 \\ -0.5 & 1 & -0.5 & x \\ x & -0.5 & 1 & -0.5 \\ -0.5 & x & -0.5 & 1 \end{vmatrix}$$

must hold according to proposition 8. This determinantal equation reduces to

$$0 \;=\; x\,(1-x)^2\,(2+x)$$

The roots corresponding to a real angle are $x = 1$ and $x = 0$. The former implies parallel opposite sides and reduces the rank of the polygon to two, while the later leaves non-zero 3×3 minors inside the determinant, a condition which must be fulfilled to have a full rank polygonal simplex of dimension three. Thus $x = 0$ is the only possible solution. Employing the projection rule (9) would have been more effective, because it says $1+\cos\alpha_{12}+\cos\alpha_{13}+\cos\alpha_{14} = 0$ and we know that $\cos\alpha_{12} + \cos\alpha_{14} = -1$. Hence, the result is immediate.

We can also rotate the whole polygon by means of an orthogonal transformation R applied to the 3×4 matrix $|w_1, w_2, w_3, w_4|$. The center of the rotated polygon is still the same because, due to isometry, the distances of the corner points to the origin remain unchanged. To change the center, in other words, to translate the polygon parallel to itself, it is sufficient to know one generating vector, which must not necessarily be the same as the generating vectors we started from. Assume that instead of v_3 we know u_3, then the new co-ordinates can be obtained by applying (3) in the inverse direction, namely $w_3 + u_3 = u_4$ leading to $w_4 + u_4 = u_1$ and so on. All these operations are isometric, thus safeguarding lengths and angles. **We may once again stress, that the easy detour to the algebraic vectors of a hyperpolygon may save n descriptive parameters in V_n related to the location of the origin of the problem, which has in a certain sense been blurred by the transition to the algebraic vectors. Saving the $0.5n(n+1)$ parameters for the rotation of the reference system is a more subtle affair, because it has to be performed at the level of the generating vectors. This is normally possible if problem specific (preferably known) vectors can be included in a vector space base system.** In fact, many well known solutions in applied mathematics related to gravitation and electromagnetism, already derived in the eighteenth or nineteenth century, do this more or less automatically, because linear algebra and its modern vector theory together with todays' computational power were not available. The, at first glance, more general problem characterizations one nowadays often finds in practice, are not seldom more intricate but often less effective.

From a didactical point of view we can imagine generalizations of almost all simple polygons in dimension two adapted to dimension three, thinking at diamonds, parallelograms, trapezoids, squares and rectangles. It gets slightly more interesting if we look for regular hyperpolygons, which we would like to define as figures having equal side lengths and equal angles at all corners. In this section we have already looked at the regular polygonal simplex or regular quadrangle in dimension three replacing the equilateral triangle in dimension two. The generalization of the common square to dimension three requires three pairs of orthogonal sides, adding up to an hyperhexagon. Remarkable is the fact that we can imagine two different versions of this hypersquare as shown in Fig. 5.

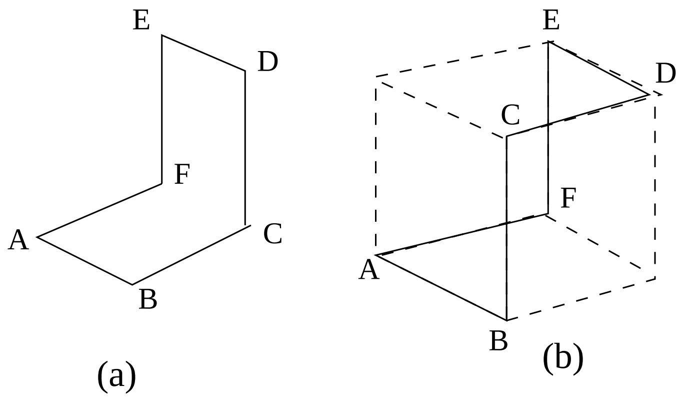

Fig 5. two types of regular hexagones in dimension three

They essentially differ by the degree of symmetry, where (b) is the more symmetrical one. We still miss the pentagon situated in between the hexagon and the quadrangle.

The regular hyperpentagon in dimension three consists of the simple concatenation of any five edges of the regular hexahedron. The latter can be considered to be constructed by gluing together two regular simplices of the same size. Obviously, the regular polytopes are a source of largely predefined regular hyperpolygons. An analysis into the existence of regular hyperpolygons beyond the link to the handful existing regular polyhedra is out of the scope of the present note.

REFERENCES

1 L. Fraiture and K. Spindler,'Spherical goniometry and Attitude Determination', Elemente der Mathematik, Vol. 57, pp. 137-157, 2002.

2 L. Fraiture,'Attitude Estimation with GPS-like Measurements', J. of the Astronautical Sciences, Vol. 54, pp. 595-617, 2006.

3 M.Berger, Geometry I, translated from the French by M. Cole and S. Levy,Springer-Verlag,second corrected printing, New York, 1994.

4 F.R. Gantmacher, Matrizen Theorie, Lizenz Ausgabe Springer-Verlag, pp. 258-266, New york 1986.

5 L. Euler,'Formulae Generales pro Translatione Quacumque Corporum Rigidorum.' Novi Commentarii Acad. Sci. Petropolitanae, Vol. 20, pp. 189-207, 1775. Alternatively, L. Euleri Opera Omnia, second series Vol. 9, pp. 84-98, Basel 1968.

6 H. Cheng and K.C. Gupta,'An Historical Note on Finite Rotations',Trans. of the ASME, J. of Applied Mechanics,Vol. 56, pp. 139-145, 1989.

7 L. Fraiture,'Propagation Algorithms for Attitude-Related Ambiguity Resolution', GPS Solutions, Vol. 6, No. 4,pp. 219-228, 2002.