NOTE 4

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DIRECTION SEPARATION IN HIGHER DIMENSIONS

Luc Fraiture†

INTRODUCTION

The subject of this note derives from the need we have had to count the occurrences of random directions with the purpose to verify their distribution in particular areas along the unit circle and on unit spheres up to dimension four. The questions we ask are of a practical nature. One is the search for a subdivision of a spherical surface to identify preferred zones. An other simple question concerns the vicinity, for instance 'which is the co-ordinate axis a particular direction is closest to?'. The actual goal of this note is twofold. First we want to introduce known mathematical insights which are complementary to the introduction of our notes 8 and 10. Second we wish to gain some insights in verification possibilities for random direction simulation in dimension four, namely there where our pictorial imagination does no longer support us. What we then do with these directions or how we generate them is a matter of statistics on the sphere for which an extensive literature exists. In this respect we refer to the book by Mardia and Jupp[1] and to our Notes 8 and 10 which deal with the generation of random directions in limited geometries on the sphere. The latter did, as far as we know, not get yet any particular attention.

DIRECTIONS AND VICINITY

Let us first agree that a direction is a unit vector v in an Euclidean vector space of dimension n denoted by V_n. The components of such a direction v are the Cartesian co-ordinates $x_1, \ldots, x_n$. We further introduce the orthonormal base vectors $u_1, \ldots, u_n$ parallel to the positive co-ordinate axes. The latter vectors are important, because they also are directions. The locus of the end points of all these directions is either a unit circle in the plane or a unit sphere in V_n.

The measure of vicinity between two directions v_1 and v_2 in V_n is the angle θ_{12} between these directions. To this aim we have the inner product at our disposition, namely:

$$< v_1 \cdot v_2 >= \cos\theta_{12} \tag{1}$$

algebraically defined by

$$< v_1 \cdot v_2 >= \sum_{i=1}^{n} x_{1i}\, x_{2i} \tag{2}$$

employing the explicit unit-vector co-ordinate representation $v_k = |x_{k1}, \ldots, x_{kn}|'$, where the accent denotes transposition. The closest directions among two unit vectors out of a collection of m vectors $(v_1, \ldots, v_m) \in V_n$ are those with the largest value of $\cos\theta_{ij}$ with

† Private note; address: Luc Fraiture, Lucasweg 6, D-64287 Darmstadt, Germany, e-mail: lucfraiture@web.de

$1 \leq i \neq j \leq m$. The smallest cosines (close to -1) are usually considered to apply to directions which are said to be *axially* close to each other. We can be more precise in saying that the vector v_k out of the collection $(v_1, \ldots, v_m)$ closest to the co-ordinate axis u_ℓ is found from

$$x_{k\ell} = \sup_{1 \leq j \leq m} x_{j\ell} \tag{3}$$

But this may not necessarily be satisfactory, because even if $x_{k\ell}$ is the largest ℓ co-ordinate in the collection scrutinized, the vector v_k can still be almost 90^0 away from the $\ell-$axis, as we will see later. The direction v_k closest to v_ℓ is obtained by the comparison of the different values of $\theta_{j\ell}$, or formally:

$$v_k = \sup_{j \neq \ell} \cos \theta_{j\ell} \tag{4}$$

To go beyond these trivial vicinity criteria we have to focus on the geometry of spheres mathematically denoted by the symbol $\mathrm{S}^{\mathrm{n}-1}$ in spaces of dimension n.

QUADRANTS AND ORTHOGONANTS

To describe the boundaries inside which directions can be confined, we will make use of the geometry of the spherical surfaces and we logically start with the circle. The simplest subdivision of the circle consists of the four *quadrants* bounded by the directions of the co-ordinate axes: u_1 and u_2, u_2 and $-u_1$, $-u_1$ and $-u_2$, $-u_2$ and u_1. Also in this case, checking in which quadrant a direction is located, is trivial. One just has to verify the sign its co-ordinates.

In dimension three we add one dimension to the circle and the previous subdivision is multiplied by two to take the two directions $+u_3$ and $-u_3$ into account. This yields eight separate surfaces, one of which is depicted in figure 1. A hemisphere has still four quadrants, but for the full sphere this name is no longer adequate. In view of what happens

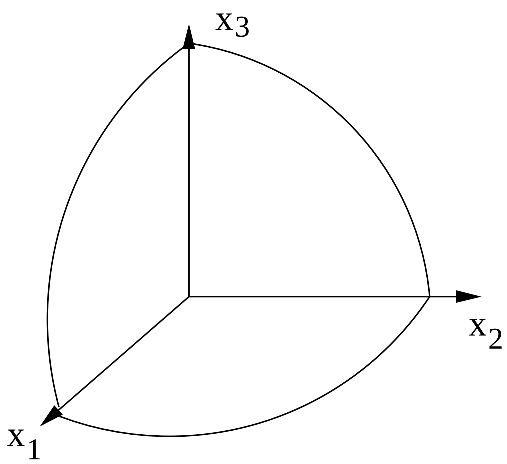

Figure 1. An orthogonant in dimension three

with such a partitioning of the sphere in higher dimensions we propose to use the name *orthogonant* for these 2^n surface units, to stress that each of these equal-shaped areas are bounded and spanned by a complete basis of n orthonormal vectors.

If we wish to proceed any further we have to complement the Cartesian co-ordinates with *spherical co-ordinates*. The name spherical co-ordinates is the generic name for all $(n-1)$-dimensional manifolds representing complete spherical surfaces in $\mathbb{R}^n$ just by the use of trigonometric functions of angles. The *polar co-ordinates* are the spherical co-ordinates for which there is a pole, which corresponds to a preferred direction coincident with one Cartesian co-ordinate axis. We comply with the widespread convention to select this axis to be $u_n \in V_n$. Further, x_n, the last component of the unit vector $v \in V_n$, is set equal to the cosine of a colatitude θ, or $x_n = \cos\theta$, independently from the other spherical co-ordinates of the manifold. Polar co-ordinates are unavoidable in odd dimensional vector spaces, in contrast to even dimensions where one can chose between polar and non-polar manifolds. For we need the surface of the complete spheres and of spherical caps, we wish to briefly review hereafter the way to formulate and integrate the corresponding known integrals.

DEFINITE INTEGRALS ON SPHERES

What we propose to do in this section, is best explained by first giving the example of dimension three. In that case the polar co-ordinates are:

$$x_1 = \cos\alpha\sin\theta, \qquad x_2 = \sin\alpha\sin\theta, \qquad x_3 = \cos\theta$$

We are primarily interested in the integral yielding the value of the area of a spherical or polar cap. In dimension three this corresponds to

$$\int_{\alpha=0}^{2\pi}\int_{\theta=0}^{\theta_0} dS \;=\; \int_{\alpha=0}^{2\pi}\int_{\theta=0}^{\theta_0} \sin\theta\, d\alpha\, d\theta \;=\; 2\pi\,(1 \,-\, \cos\theta_0) \tag{5}$$

This also provides the surface 4π rad^2 of the complete sphere by setting $\theta_0 = \pi$ in the previous integral. The differential surface element dS can be found from basic differential geometry in the way explained hereafter.

Consider the orthogonal Cartesian co-ordinates $(x_1, \ldots x_n)$ and transform these co-ordinates into the curvilinear co-ordinates $(y_1, \ldots y_\ell)$ by means of the following transformation:

$$\begin{aligned} x_1 \;&= X_1(y_1, \ldots y_\ell) \\ &- - - \\ x_n \;&= X_n(y_1, \ldots y_\ell) \end{aligned}$$

where $y_1, \ldots, y_\ell$ are supposed to be mutually independent and $\ell = n$ for the parameterization of a ball (or a circular disc) and $\ell = n-1$ for a sphere (or the circumference of a circle). The differentials of the Cartesian co-ordinates are mutually orthogonal infinitesimal line elements equal to:

$$dx_j \;=\; \sum_{i=1}^{\ell} \frac{\partial X_j}{\partial y_i}\, dy_i \qquad (j \,=\, 1, \ldots, n)$$

And because the world of our considerations is Euclidean, the square of the total resulting infinitesimal displacement $ds^2 \;=\; dx_1^2 + \ldots + dx_n^2$ must remain the same in the transformed system. Thus

$$ds^2 = \sum_m^n dx_m^2 = \sum_i^\ell \sum_j^\ell g_{ij}\, dy_i\, dy_j \tag{6}$$

with the short hands

$$g_{ij} = \sum_m^\ell \frac{\partial X_m}{\partial y_i} \frac{\partial X_m}{\partial y_j} \tag{7}$$

which are called *the metric coefficients.* All we need for our purposes are the metric coefficients g_{ii}, because the polar/spherical co-ordinates of circles and spheres are such that the metric coefficients g_{ij} all vanish when $i \neq j$. If more background is desired, we refer to the very readable chapters VII and VIII of the classical book by A.P. Wills[2]. To make the computation of the metric factors – appearing in the infinitesimal volume and surface elements – plausible, we consider, as example, the polar co-ordinates y_1 and y_2 in the plane defined by:

$$x_1 = y_2 \sin y_1 = r \sin\theta, \qquad x_2 = y_2 \cos y_1 = r \cos\theta$$

The computation of ds^2 as a function of y_1 and y_2 then yields:

$$dx_1^2 + dx_2^2 = y_2^2\, dy_1^2 + dy_2^2 = r^2\, d\theta^2 + dr^2$$

and the metrical coefficients $g_{11} = g_{\theta\theta} = y_2^2 = r^2$ and $g_{22} = g_{rr} = 1$. For a circle with radius r_0 the circumference is given by $\int_0^{2\pi} \sqrt{g_{\theta\theta}}\, d\theta$ and the surface is obtained from $\int_0^{2\pi} \int_0^{r_0} \sqrt{g_{\theta\theta}\, g_{rr}}\, d\theta\, dr$. The metric factor for an infinitesimal arc element hereby appears to be equal to $\sqrt{g_{\theta\theta}}$ and for a circle-surface element we get $\sqrt{g_{\theta\theta}\, g_{rr}}$. Referring to the spherical co-ordinates defined hereafter the metric factor for the volume element of a three dimensional ball is $\sqrt{g_{\alpha\alpha}\, g_{\theta\theta}\, g_{rr}}$, while for the surface element on S^2 it is equal to $\sqrt{g_{\alpha\alpha}\, g_{\theta\theta}}$. Extrapolation to spheres of higher dimensions follows the same pattern.

The polar co-ordinates $(\alpha_1, \ldots, \theta)$ on unit spheres in dimension $n \leq 4$ have important practical applications, and their correspondence with orthogonal Cartesian co-ordinates is displayed in the overview given hereafter.

	S^1	S^2	S^3
x_1	$= \sin\theta$	$= \cos\alpha \sin\theta$	$= \cos\alpha_1 \cos\alpha_2 \sin\theta$
x_2	$= \cos\theta$	$= \sin\alpha \sin\theta$	$= \cos\alpha_1 \sin\alpha_2 \sin\theta$
x_3		$= \cos\theta$	$= \sin\alpha_1 \sin\theta$
x_4			$= \cos\theta$
dS	$= d\theta$	$= \sin\theta\, d\alpha\, d\theta$	$= \sin\alpha_1 \sin^2\theta\; d\alpha_1 d\alpha_2\, d\theta$

We now can compute the area of a polar spherical cap on S^3. We have

$$\int_{\alpha_2=0}^{2\pi} \int_{\alpha_1=0}^{\pi} \int_{\theta=0}^{\theta_0} \sin\alpha_1\, \sin^2\theta\, d\alpha_1\, d\alpha_2\, d\theta$$

$$= 2\pi \int_{\alpha_1=0}^{\pi} \sin\alpha_1\, d\alpha_1 \int_{\theta=0}^{\theta_0} \sin^2\theta\, d\theta = 2\pi\, (\theta_0 - 0.5 \sin 2\theta_0) \tag{8}$$

Again, this yields the surface $2\pi^2$ rad^3 of the complete sphere by setting $\theta_0 = \pi$ in the previous integral. Especially this case shows that a spherical cap of angular radius θ_0 around the pole is a hyper-cone comprising all direction which are at an angular distance of the pole comprised between zero and θ_0 independent of what happens in the other directions. The formulae for the area of the cap as a function of θ_0 differs, of course, for every dimension, as (6) and (8) demonstrate. In practice any direction v_0 can be considered to be a 'pole' and apply (5) and (7) where θ is simply the angular distance from v_0.

In many applications the polar axis plays a fundamental role and leads to considerations about symmetry around this axis. Before we can address this aspect we need the generalized meridian concept introduced in the next section.

FURTHER CARTESIAN BASED PROPERTIES OF DIRECTIONS

We further introduce geometrically oriented notions and properties, which all derive from the definition of a meridian on a sphere with a pole v_0 parallel to $u_n \in V_n$ in dimension $n = 3$ or higher.

Definitions

Let $v^{\perp}$ be a unit vector such that $v^{\perp} \cdot v_0 = 0$, then the locus of all unit vectors $v = av^{\perp} + bv_0$ for any a and b with $a^2 + b^2 = 1$ is called a *meridian* through the pole v_0.

There are $(n-1)$ mutually orthogonal meridians at a pole on S^{n-1} and they only meet again at the anti pole where they are, of course, again orthogonal. By a *base meridian* $\mathcal{M}_j$ we mean a meridian generated by u_j and v_0.

Consistent with the terminology just proposed we will call *equator plane* of the sphere S^{n-1}, that part of the subspace orthogonal to u_n which is fully contained within the sphere S^{n-1}. Hence, the locus of all unit vectors in the equator plane is equivalent to a S^{n-2} (unit) sphere.

If a direction v is parametrized by signed arcs on base meridians $\mathcal{M}_j$ at the pole, signed arc lengths will be denoted by ϵ_j and referred to as *surface pseudo co-ordinates* of a direction.

The actual projection of ϵ_j on the co-ordinate axis u_j with $j \neq n$ is $x_j = \sin \epsilon_j$, which paradoxically is a direction cosine. Let us first define an arbitrary unit vector v_e by means of its Cartesian components $(x_{e1}, \ldots, x_{en})$ where the last component is along u_n by convention. To verify the nature of the co-ordinates ϵ_j, we start from the transformation:

$$x_1 = \sin \epsilon_1, \qquad x_2 = \sin \epsilon_2, \qquad x_3 = (1 - \sin^2 \epsilon_1 - \sin^2 \epsilon_2)^{-1/2}$$

valid for dimension three. Writing Q for $1 - \sin^2 \epsilon_1 - \sin^2 \epsilon_2$ the squared infinitesimal line element (6) appears to be:

$$ds^2 = Q^{-1} (\cos^2 \epsilon_1 \cos^2 \epsilon_2 \, d\epsilon_1^2 + \cos^2 \epsilon_1 \cos^2 \epsilon_2 \, d\epsilon_2^2 + 0.5 \sin 2\epsilon_1 \sin 2\epsilon_2 \, d\epsilon_1 \, d\epsilon_2)$$

The only points where ϵ_1 and ϵ_2 behave like orthogonal Cartesian co-ordinates are the poles. Consequently, a relation like $\epsilon_1^2 + \epsilon_2^2 = \theta^2$ can only be accepted if ϵ_1 and ϵ_2 are very small. Such approximations can be found in geometrical optics for narrow field of views[3]. We can thus only employ the pseudo co-ordinates, if we go via the Cartesian co-ordinates and stay away from the equator where x_3 gets imaginary.

Due to the fact that v_e is a unit vector we have $\sum_{i=1}^{n} x_{ei}^2 = 1$ or equivalently $\sum_1^{n-1} x_{ei}^2 = 1 - x_{en}^2$, but $x_{ei} = \sin\epsilon_i$ for $i < n$, which makes that

$$\sin^2\theta \;=\; \sum_{i=1}^{n-1} \sin^2\epsilon_i \tag{9}$$

This at first glance strange equation is obvious if we realise that θ and $\pi/2 - \epsilon_i$ are both colatitudes. We can further stress the 'pseudo' character of the ϵ_i components by noticing that

$$v_e \;=\; \sum_{i=1}^{n-1} \sin\epsilon_i\, u_i \;\pm\; \sin\theta\, u_n \tag{10}$$

All this brought us to the idea that the simulation of the $n-1$ mutually independent uniformly distributed random pseudo co-ordinates directly on each base meridian must lead to uniform distributed points on rectangles on the sphere. The projection of these points on the equator yields a direction provided $\sin\theta$ is real. This condition further requires that

$$0 \le \sum_{i=1}^{n-1} \sin^2\epsilon_i \;\le\; 1 \tag{11}$$

is satisfied. If (11) applies we have found a random direction vector corresponding to a uniform distribution on the sphere also in higher dimensions. Details of this method are provided in our Note 10.

Let us once more consider (9), but this time in the light of sample property verification. First of all, we have to assume that all unit vectors d_k with $1 << k \le m$ of a given sample are contained in one hemisphere with respect to the polar axis, excluding the equator itself. Thereby all components $(d_k)_n$ on the polar axis u_n have the same sign. In this way we avoid the inclusion of part of the geometry where pseudo co-ordinates do not exist. Then, taking the mathematical expectation of (9) means

$$\overline{\sin^2\theta} \;=\; \sum_{k=1}^{m} \sum_{i=1}^{n-1} \overline{(d_k)_i^2} \tag{12}$$

This is particularly useful if there are enough data to compute the empirical covariance matrix whose trace can then be compared with the theoretical expectation of the polar colatitude ϵ whose value should be similar to $\theta = \arcsin[(\overline{\sin^2\theta})^{1/2}]$. For modest and small samples we refer to the book of Mardia and Jupp[1], which is in any case worth consulting when performing experimental statistics on S^2.

GEOMETRY AROUND THE POLAR AXES

For we deal with the count of occurrences, symmetry must be understood as a property of the probability of these direction occurrences, and more specifically of a **p**robability **d**istribution **f**unction (pdf). For the purpose of discussing symmetry, pdf's can be expressed as a function of the different manifolds we have introduced, thus, for instance, by

$\Phi(x_1, \ldots, x_{n-1})$. We do not wish to detail here any other properties of pdf's, except their potential symmetry aspects. In this respect we differentiate between:
- *Axial Symmetry* (AS). This symmetry implies that

$$\Phi(x_1, \ldots, x_{n-1}) \;=\; \Phi(-x_1, \ldots, -x_{n-1}) \tag{13}$$

- *Rotational symmetry* (RS). This symmetry implies that

$$\frac{\partial \Phi(\alpha_1, \ldots, \alpha_{n-1}, \theta)}{\partial \alpha_i} \;=\; 0 \qquad \text{for any } i \leq n-1 \tag{14}$$

- *Rotational Uniformity* (RU). RU assumes that the probability to have direction occurrences is the same on any meridian (and its vicinity) issuing from the pole, independent of the probability distribution inside this neighborhood. This implies the following integral property for any $i \neq j$:

$$\int_{\epsilon_i=0}^{\pi} \Phi(\sin\epsilon_1, \ldots, \sin\epsilon_{n-1})\, \sqrt{g_{ii}}\, d\epsilon_i \;=\; \int_{\epsilon_j=0}^{\pi} \Phi(\sin\epsilon_1, \ldots, \sin\epsilon_{n-1}) \sqrt{g_{jj}} d\epsilon_j \tag{15}$$

as far as (11) is satisfied. This statistical symmetry concept, which we did not meet in the technical literature, may best be explained by the example of an unimodal error distribution (this is a distribution which, in the present terminology, has a single probability maximum in the direction of the pole) which is assumed to be AS. If moreover the corresponding pdf is RU, it is equally probable to find random directions near meridians where large errors are more probable as well as near meridians where small errors are more likely. To quote another example, we consider a flat square (having the pole or mode in its center) filled with uniformly and randomly distributed co-ordinate pairs. This distribution is not RU, because one can expect more random points along the diagonals than along lines through the center and parallel to the sides. The RU concept can be useful when it points to the smaller/larger probability to be in (un)favorable directions in a control problem, for instance.

Especially the detection and checking of rotational uniformity in higher dimensions by means of counting requires further insight about the idea of proximity with respect to meridians. In our imagination we are inclined to see the confluent meridians on the three dimensional globe as something which could re-occur in higher dimensions. In dimension three, however, the angle between two meridians is a dihedral angle, an entity which does not exist in dimension four or higher. Otherwise the sum of such generalized dihedral angles among all the base meridians and their negative sides would add up to an angle of $2^{n-1}(\pi/2)$, which is impossible, if the definition of angles is based on an inner product, as is the case here. We thus have to have recourse to more basic properties. Therefore, we start by considering the normalized orthogonal projection v_p of an arbitrary unit vector $v(x_1, \ldots, x_n) \in S^{n-1}$ onto the equator not bothering about the fact that $v_p \in S^{n-2}$. In this way we obtain the following direction cosines of v_p in the equator:

$$\cos\omega_i \;=\; \frac{x_i}{\sqrt{1 \,-\, x_n^2}} \;=\; \frac{\sin\epsilon_i}{\sqrt{1 \,-\, x_n^2}} \qquad (i = 1, \ldots, n-1) \tag{16}$$

because any vector in the equator is orthogonal to u_n. But the direction cosines $x_{pi} = \cos\omega_i$ of v_p must satisfy:

$$\sum_{i=1}^{n-1} \cos^2\omega_i \;=\; 1 \tag{17}$$

because we are dealing with a unit vector. Consequently, we can state that

$$\sum_{i\neq j}^{n-1} \cos^2\omega_i \;=\; \sin^2\omega_j \tag{18}$$

These relations allow to practically resolve the question about the closest proximity to a Cartesian co-ordinate axis on the basis of the following simple but, in this context, nevertheless important theorem.
THEOREM. If a direction $v \in V_n$ makes an angle ω_i of less than $\pm 45^0$ or equivalently $\pm\pi/4$ rad with the co-ordinate axis direction u_i, then the angles v can make with any direction orthogonal to u_i are all larger than $\pi/4$ in absolute value for $2 \leq n$.
PROOF. By reconsidering (18) we can say that

$$\cos^2\,\omega_i \;\leq\; \sin^2\omega_j \qquad (i \neq j)$$

where ω_j, the object of the theorem, is the angle which $v \in V_{n-1}$ makes with u_j, while ω_i is the angle between v and an arbitrary base vector u_i with $i \neq j$. Restricting our considerations to the first quadrant, previous inequality means that

$$\frac{\pi}{2} - \omega_i \;\leq\; \omega_j$$

If now $\omega_j \; < \pi/4$ we have $\pi/2 - \omega_i \; < \pi/4$ and thus $\pi/4 \; < \; \omega_i$, as claimed. For this argument can be repeated for any angle $\omega_i \neq \omega_j$, the theorem is verified. ■

The previous theorem applies to the projections of a direction onto two arbirtrary mutually orthogonal vectors in any dimension, because we can always add a third vector orthogonal to the two first and this latter vector can be considered to be the pole in a higher dimension.

ISOLATING PARTS ON SPHERES IN HIGHER DIMENSIONS

There are two major areas of practical interest. The first involves complete spheres for which a large number of efficient statistical analysis criteria and algorithms are available in dimension two and three for small and medium size direction samples[1]. In dimension four and certainly in higher dimensions the available techniques are limited. The second area of interest deals with limited parts of spherical surfaces centered around a pole, with practical applications in simulations[4] and the statistics of direction estimation errors also in the case of dimension four.

Distribution of Spherical Caps on the Sphere

As a consequence of the previous theorem, one can add a spherical cap with an arc radius of $\pi/4$ rad at the end of each of the $2n$ positive and negative ends of co-ordinate axis directions. This provides an ability to subdivide a sample of directions into sets being closest to a particular positive and/or negative co-ordinate axis. In dimension two this subdivision covers the complete circle, because the proposed caps are in fact arcs of 90^0 length. In dimension three the caps of 45^0 arc radius do not longer cover the sphere. This is shown in Fig. 2 with a three dimensional orthogonant containing three mutually tangent quarter-caps. Not covered by these caps is an improper spherical triangle ABC enclosed between the caps. The center of this triangle denoted by N in the figure, which will further be called 'neutral point', has its three direction cosines all equal to $1/\sqrt{3}$ corresponding to an angle of $\theta_{*n} = \theta_{*3} \approx 54.763^0$. By inspecting the figure more closely we see that the caps do not contain *all* the directions closest to a co-ordinate axis. For x_3 this would be the spherical quadrangle X3ANC composed of two (canonical) spherical triangles whose arcs and dihedral angles are known except the arcs AN=NC, which can be computed by means of the cosine rule. This yields $\cos \mathrm{AN} = \sqrt{2}/\sqrt{3}$. Gluing all these quadrangles together at the six co-odinate axis ends yields a regular spherical cube composed of six equal spherical squares known in geometry. Applying (3) to the absolute values of the co-ordinates of the directions

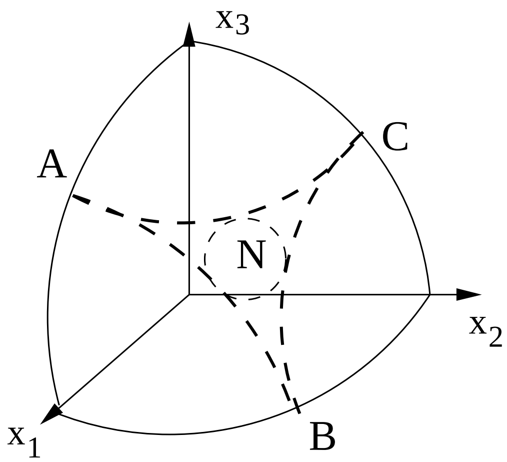

Figure 2. Spherical caps and the neutral area n in dimension three

yields the closest co-ordinate axis, but in dimension three this can already be as distant as θ_{*3}. Depending on the problem at hand, we may require a refined answer by checking the exclusive presence in a cap, but this ability ends with an arc-radius bound by $\pi/4$ in any dimension. In practice, we may also verify whether a direction, which was has not been identified to fit into one of the 45^0-caps, is located inside the spherical cap around the neutral point N with an arc radius of $54.763^0 - 45^0$. There could be two practical reasons for this. First, accumulation of directions at symmetry points like N may be desired or, on the contrary, may result from undesired simulation implementation features. Second, the verification with a cap around the neutral point N is still easy to implement in higher dimensions.

In dimension three, 93.6% of the total surface of S^2 is covered by the sum of the 45^0 caps and the caps around the neutral point, leaving little uncertainty with respect to the identification of distribution deviations. One can achieve 100% coverage if one subdivides S^2 in its six equal great circle spherical squares whose corners coincide with the eight neutral points (a spherical hexahedron). The question is whether the added value of 6.4% of coverage is worth the trigonometrical analysis and coding.

The efficiency of coverage verification rapidly decreases with increasing dimensions. This decrease is compensated by the fact that we can, in principle, compute the size of all areas involved for all finite dimensions. To understand the way these areas can be evaluated, we have to get an idea of the structure of an n-dimensional orthogonant starting from Fig. 2. Let us take the orthogonant coinciding with all positive co-ordinate directions. In the n corners, if we are still allowed to call them so, there are n caps of 45^0 and each of these caps is tangent to all the others. Between these caps there is still a neutral point whose direction cosines are all equal $1/\sqrt{n}$. In between these caps the improper triangle has been replaced by a surface which we could imagine to be a spider with n legs having a neutral point in its center. Thus, the surface of the total sphere minus the surface of the $2n$ 45^0 caps yield the surface of the 2^n spiders. The area of a single spider contains the n contributions of the n single co-ordinate directions (by the selection of the orthogonant here all positive). In dimension four the coverage with eight 45 degree caps and the sixteen caps around the neutral points in each orthogonant approximately represents 78,6% of the total spherical surface.

Combining Caps and Colatitudes

If we consider distributions of directions around a pole, the selection of a pole is already a reason to look into the distribution as a function of the colatitude. To that

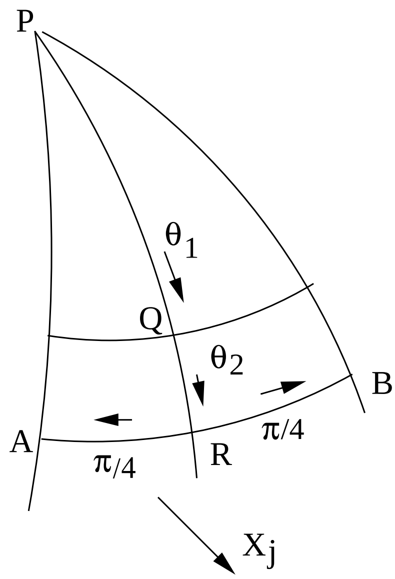

Figure 3. The combination of caps with co−ordinate axis proximity

purpose one may conceive a number of (hyper)rings around the pole for which the surface area is easily computed based on integrals formulated with polar co-ordinates. Once we know that the colatitude θ of a direction v is in an area bounded by θ_1 and θ_2 we are left with the question: 'where is it located around the pole?'. The projection of of v into the equator allows the identification of the closest co-ordinate axis direction, as explained before. This geometrical situation is depicted in Fig. 3. The subdivision of the equator in 45^0 caps only covers all directions if we consider polar co-ordinates on S^2. If we deal with polar co-ordinates on S^3, the equator is a S^2 sphere and then the technique suggested in Fig. 3 involves neutral points and neutral caps, as explained before. By increasing dimensions this neutral area increase, and this is the theme of the last section.

THE PSEUDO-ORTHOGONALITY PHENOMENON

The subject of this section is of a pure mathematical nature and deals with a curiosity which may suggest some philosophical questions.

Summarizing what we have said in the previous sections, we know that an orthogonant on S^{n-1} is bound by n end directions of orthogonal co-ordinate axis (positive or negative). At each of these n orthogonal directions we can attach a 45^0 spherical cap and each of these caps is tangent to each of the $n-1$ other caps bordering the orthogonant under consideration. In each orthogonant there is a neutral point equidistant to all n orthogonal axes bordering that orthogonant. The corresponding angular distance ϕ of this neutral direction to any of the co-ordinate axes of the orthogonant is equal to $\arccos(1/\sqrt{n})$. We are for each dimension able to define caps around these neutral directions, whose arc-radius is equal to $\phi - \pi/4$ which are tangent inside an orthogonant to each of the 45^0 caps around the co-ordinate axes. The point is that $\lim_{n\to\infty} \phi = \pi/2$. By approaching this limit for a given still finite value of n we get the impression that the 2^n neutral points are co-ordinate axes as well, and if this were true we had at once dimension 2^{n+1}. But this brings us closer the limit and if we could take the first step in a good approximation, the next applies to an even better approximation. In principle this process, once started and brought to its end finishes with a mathematical explosion of the dimensionality.

In practice we will not suffer from this effect. The weak point of the argument is the assumption of *'for a still finite value . . . in a good approximation'*, because the explosion which would take place could not be reverted to the original situation, where $1/\sqrt{n}$ could not be restored to the original value still a minute but nevetheless finite ε away from the limit. In fact this $1/\sqrt{n}$ is not an approximation but an exact number. All this is reminiscent of the law of large numbers in probability.

REFERENCES

1 Mardia K.V., Jupp P.E.,Directional statistics, John Wiley & Sons, New York, 2000.

2 Wills A.P.,Vector Analysis with an Introduction to Tensor Analysis, Dover Publications, New York, 1958.

3 Weinberger M.,'Statistics of Spacecraft Pointing and Measurement Error Budgets', Journal of Guidance Control and Dynamics, Vol. 17, No.1, 1994, pp. 55-61.