NOTE 6

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DIRECT SOLUTIONS OF PARTICULAR SYSTEMS OF THREE QUADRATIC EQUATIONS

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Abstract. We present solutions for a collection of particular systems of three quadratic equations with three unknowns which involve squares and/or bilinear terms in all unknowns in each equation. All these particular systems allow simplifications, avoiding the analytical resolution of a fourth degree polynomial as part of the direct elimination process. The more important cases are the systems without linear unknowns, the systems where two equations have no linear unknowns, the systems void of bilinear unknown combinations but with linear unknowns and finally the systems without squares. Complementary to the algorithms a criterion is derived which shows whether the sum of two indefinite three dimensional matrices can be made positive definite.

1. INTRODUCTION

The properties of single algebraic quadratic expressions have extensively been studied in the nineteenth century by a large number of eminent mathematician like Hesse, Sylvester, Jacobi and Hermite, without being exhaustive. They laid the foundation for the further study of bilinear and quadratic expressions in linear algebra in the early twentieth century. What we need for our purposes from this vast theoretical body is quite minute and is introduced in the next two sections. Some practical information about positive definiteness is added in the last section.

If we make the step from a single expression to systems of quadratic equations, the potential for a theoretical expansion is modest. Nevertheless, the properties of the square symmetric or Hermitian matrices directly lead to a simplification which can be exploited if there are at least two equations in a quadratic system (of any dimension) which are void of linear unknowns. This is addressed in the classical book of Courant and Hilbert[1] as well as by Gantmacher[2], who both give an extended survey of the handling of quadratic forms. Nevertheless, all this does not lead to an analytical solution in systems of dimension as low as three. Analytical solutions were the challenges for the gifted mathematicians up to the end of the nineteenth century. The applied mathematical processes implied to solve a quadratic system in dimension three, must have been clear during Hermite's live, but they

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were then still so cumbersome that practical need would have been the only real motivation to report the algorithms involved for finding the solutions. As far as we know, this is still true for the direct resolution of such a general three-dimensional quadratic system with real coefficients. Therefore, the practitioner finds little about this specific problem, a fact which had motivated us some forty years ago to look for algorithmic simplifications. We only succeeded to derive such simplifications for a number of particular cases, limited to dimension three, which in the end appear to be straight forward exercises of linear and conventional algebra. Consequently, they are of little interest in pure mathematics, but in practice, electronic computing changed our world so fundamentally that knowledge about the way to obtain complete numerical solutions has gained considerable importance.

The analytical geometry of quadrics is intimately linked to the properties of three dimensional quadratic forms. But although enormous progress has been made in the general geometry of quadratic forms and their combination, we could not find something which could be of general practical help in resolving system of three quadratic equations in three unknowns[3]. The few very general aspects related to quadrics which we may mention later on, can be considered to be well known.

2. TRANSFORMATIONS OF A SINGLE QUADRATIC EQUATION

In all what follows we will work with a three dimensional representation. Hence, the vector of unknowns $\mathbf{x}$ contains the components (co-ordinates) x, y and z. The vector $\mathbf{x}$ will be subject to different sequences of linear transformations and these transformed vectors will be written either as $\mathbf{u}$, $\mathbf{v}$ or $\mathbf{w}$ and their components will be identified by subscripts to the lower case non-bold character corresponding to the vector representation.

Let H be a 3×3 real symmetrical matrix and $\mathbf{b}$ a vector with the components (b_x, b_y, b_z) then a quadratic equation can be represented by

$$\begin{aligned} c = \mathbf{x}' H \mathbf{x} + 2\,\mathbf{b}'\mathbf{x} = h_{11}\,x^2 + 2\,h_{12}\,xy + 2\,h_{13}\,xz \\ +h_{22}\,y^2 + 2\,h_{23}\,yz + h_{33}\,z^2 + 2b_x\,x + 2b_y\,y + 2b_z\,z \end{aligned} \qquad (1a)$$

where c is a constant and an accent, added to a vector or a matrix, denotes transposition. In analytical geometry the Hessian of the previous equation is the following determinant value:

$$Hessian = \det \begin{pmatrix} h_{11} & h_{12} & h_{13} & b_x \\ h_{12} & h_{22} & h_{23} & b_y \\ h_{13} & h_{23} & h_{33} & b_z \\ b_x & b_y & b_z & -c \end{pmatrix} \qquad (1b)$$

and by discriminant one understands $\det(H)$. We will not use these quantities which are essential tools in the study of the geometrical properties of quadrics.

We first consider a translation or displacement of the co-ordinate origin by the vector $\mathbf{d}$ such that we obtain the transformed vector $\mathbf{u} = \mathbf{x} + \mathbf{d}$. To safeguard the information contained in (1) this transformation necessarily corresponds to:

$$\begin{aligned} c &= (\mathbf{u}-\mathbf{d})'\,H\,(\mathbf{u}-\mathbf{d}) \;+\; 2\,\mathbf{b}'\,(\mathbf{u}-\mathbf{d}) \\ &= \mathbf{u}'\,H\,\mathbf{u} + 2(\mathbf{b}' \;-\; \mathbf{d}'\,H)\,\mathbf{u} \;+\; \mathbf{d}'\,H\,\mathbf{d} \;-\; 2\,\mathbf{b}'\,\mathbf{d} \end{aligned} \tag{2}$$

which means that the representation of the quadratic and bilinear terms remains unchanged, but from the coefficients of the linear terms $2\mathbf{b}'$ one has to subtract $2\mathbf{d}'\,H$ and from the constant c one will deduct $(\mathbf{d}'\,H\,\mathbf{d} \;-\; 2\,\mathbf{b}'\,\mathbf{d})$.

Next we want to introduce a linear transformation of the unknowns. We represent this transformation by a non-singular 3×3 matrix T and the new unknowns by $\mathbf{v} = T\,\mathbf{x}$. Equation (1) is then equivalent to

$$\begin{aligned} c &= (\mathbf{x}'T')T'^{-1}\,H\,T^{-1}(T\mathbf{x}) \;+\; 2\,\mathbf{b}'\,T^{-1}(T\mathbf{x}) \\ &= \mathbf{v}'(T'^{-1}\,H\,T^{-1})\mathbf{v} \;+\; 2\,\mathbf{b}'\,T^{-1}\mathbf{v} \end{aligned} \tag{3}$$

If $T = A$ is a non-singular orthogonal matrix for which $A' = A^{-1}$ or equivalently $A'A = I_3$, where I_3 is a three dimensional unit matrix, one calls $A'\,HA$ a similarity transformation of H. The real symmetric matrices H are said to be Hermitian which means that this type of matrices has real eigenvalues with their mutually orthogonal eigenvectors. If some eigenvalues are not different the corresponding eigenvectors are not unique. Moreover, if H has full rank (which means "is not singular") these normalized eigenvectors can be put in the columns of A to obtain an orthogonal matrix. If this is the case, $A'\,H\,A \;=\; \Delta$ where Δ is a diagonal matrix whose diagonal elements are equal to the eigenvalues of H.

Let us now assume that T in (3) is the orthogonal matrix A just mentioned so that (3) becomes

$$c \;=\; \mathbf{v}'\,\Delta\,\mathbf{v} \;+\; 2\mathbf{b}'\,A'\,\mathbf{v} \tag{4}$$

We introduce the shorthand $\mathbf{g} \;=\; A\mathbf{b}$. Thereby the detailed algebraic transcription of (4) is equal to:

$$c \;=\; e_1\,v_1^2 \;+\; e_2\,v_2^2 \;+\; e_3\,v_3^2 \;+\; 2\,g_1\,v_1 \;+\; 2\,g_2\,v_2 \;+\; 2\,g_3\,v_3 \tag{5}$$

where e_i is the i-th real eigenvalue which is non-zero if the original matrix H is non-singular. This equation can almost trivially be reduced to a sum of squares void of linear terms. To this aim we consider the following relation:

$$\begin{aligned} e_i\,v_i^2 \;+\; 2\,g_i\,v_i \;&=\; e_i\,(v_i^2 \;+\; 2\,\frac{g_i}{e_i}\,v_i \;+\; (\frac{g_i}{e_i})^2) \;-\; \frac{g_i^2}{e_i} \\ &=\; e_i\,(v_i \;+\; \frac{g_i}{e_i})^2 \;-\; \frac{g_i^2}{e_i} \end{aligned}$$

Thus, by substituting the translated unknown $w_i = v_i + g_i/e_i$ in the previous equation, we obtain

$$c + \sum_{i=1}^{3} \frac{g_i^2}{e_i} = e_1\, w_1^2 + e_2\, w_2^2 + e_3\, w_3^2 \tag{6}$$

Such a sum of squares can be obtained for any quadratic form (of any dimension), whether homogeneous or not, a fact which had already been derived by Lagrange. The way it is presented here implies the insights Hermite already had some 150 years ago. It is important to realize that the constant on the left and the coefficients on the right are all still real provided H and $\mathbf{b}$ were real at the start.

For the sake of completeness we may mention that a full rank symmetric matrix H can be factorized into the product of a(n) (upper/lower) full rank triangular matrix S with its transpose, or $H = S'S$. We did not identify any particular advantage in employing the factorization in triangular matrices when dealing with a system of quadratic equations, whatever the special merits of such a factorization may be when considering a single quadratic form.

3. THREE QUADRATIC EQUATIONS IN A SYSTEM

Let us recall that we start from three quadratic equations in three unknowns with the general shape:

$$\Phi_i(H_i, \mathbf{b_i}, c_i) = \mathbf{x}'\, H_i\, \mathbf{x} + 2\, \mathbf{b}_i'\, \mathbf{x} - c_i = 0 \qquad (i = 1, 2, 3) \tag{7}$$

The part of these equations which ensures the quadratic nature, is the Hermitian matrix H_i. In order to be really sure that we deal with a **three** dimensional **quadratic** system we must require that the lowest rank of the linear combination $\sum_{i=1}^{3} \lambda_i\, h_i$ for whatever non-trivial values of λ is at least equal to one. Otherwise there is a non-trivial combination which leads to the linear equation

$$2\, (\sum_{i=1}^{3} \lambda_i\, \mathbf{b}_i')\mathbf{x} = \sum_{i=1}^{3} c_i$$

which allows to remove one unknown by simple substitution without affecting the quadratic nature of the resulting two quadratic equations in two unknowns.

Assuming that we have a true quadratic system in three unknowns, it is, if feasible, permitted to arrange the system so that H_1 has full rank. We make use of this fact to find and apply the linear orthogonal transformation matrix A diagonalizing H_1 by means of a similarity transformation and apply it to the other two equations to maintain consistency. Also the translation required to obtain the sum of squares without linear terms for the first equation of the system as shown in (5), is applied to the other two equations as well. It is assumed that by these transformations the new unknowns $\mathbf{w}$ are equal to $A\mathbf{x} + \mathbf{d}$, where $\mathbf{d}$ is the translation required to reduce the first equation to a sum of squares, yielding the transformed equation system represented by:

$$\Phi_1(\Delta(e_1, e_2, e_3),\, \mathbf{0},\, c_1) \;=\; e_1\, w_1^2 \,+\, e_2\, w_2^2 \,+\, e_3\, w_3^2 - c_1 \;=\; 0 \tag{8a}$$

$$\Phi_2(F^{(2)},\, \mathbf{g}^{(2)},\, c_2) \;=\; \mathbf{w}'\, F^{(2)}\, \mathbf{w} \,+\, \mathbf{g}^{(2)'}\, \mathbf{w} \,-\, c_2 \;=\; 0 \tag{8b}$$

$$\Phi_3(F^{(3)},\, \mathbf{g}^{(3)},\, c_3) \;=\; \mathbf{w}'\, F^{(3)}\, \mathbf{w} \,+\, \mathbf{g}^{(3)'}\, \mathbf{w} \,-\, c_3 \;=\; 0 \tag{8c}$$

In general (8a) is further transformed by a rescaling

$$y_i \;=\; \sqrt{|e_i|}\, w_i \tag{9}$$

If H_1 is indefinite the signs of its three eigenvalues are different and (8a) will, even after rescaling, represent an one or two bladed hyperboloid with its main axis centered at the origin. If on the other hand H_1 is positive definite, the rescaling will change the figure into a sphere (apart from c_1). The sphere is insensitive to a rotation, while hyperboloids are modified in their description as a consequence of a rotation. If we wish to manipulate the other two equations by means of rotations and leave the first unchanged we thus have to require that H_1 is positive definite at the start, which unfortunately is not always feasible. We will address this problem in the last section of this note.

The pattern of positive and negative eigenvalues of a particular Hermitian matrix cannot be modified by submitting it to a similarity transformation (due to the well known inertia of single quadratic forms). A potentially imaginary rescaling $y_i \;=\; \sqrt{e_i}\, w_i$ which one could apply if $e_i < 0$ for at least one value of $1 \leq i \leq 3$ will thus not help. Furthermore, this brings us into the realm of Hermitian vector spaces applicable to the field of complex numbers with sesquilinear inner products. Our algorithms can be adapted to cope with such a situation, but one has to take care that the symmetric matrices remain self adjoint, see for instance Wilf[4], who gives a rather succinct but clear insight into the properties of Hermitian matrices and related problems. In this note we only consider real transformations and to make the inverse transformations after having derived the numerous solutions for the last (transformed) unknown – most of which are likely to be complex – one will just employ the inverse of the real transformations which led to the (higher degree) polynomial of the surviving unknown after elimination.

4. CASES WITH ANALYTICAL SOLUTIONS

CASE I

The well known system

$$0 \;=\; \Phi_1(\Delta(e_{11}, e_{12}, e_{13}),\, \mathbf{0},\, c_1) \tag{10a}$$

$$0 \;=\; \Phi_2(\Delta(e_{21}, e_{22}, e_{23}),\, \mathbf{0},\, c_2) \tag{10b}$$

$$0 \;=\; \Phi_3(\Delta(e_{31}, e_{32}, e_{33}),\, \mathbf{0},\, c_3) \tag{10c}$$

is equivalent to a linear system with the unknowns w_1^2, w_2^2 and w_3^2 for which we obtain the unique solution $\hat{w}_1^2, \hat{w}_2^2$ and $\hat{w}_3^2$ which gives rise to the eight solution-vectors $|\pm\sqrt{\hat{w}_1^2},\, \pm\sqrt{\hat{w}_2^2},\, \pm\sqrt{\hat{w}_3^2}\,|$ provided the 3×3 matrix E with the elements e_{ij} is non-singular.

CASE II A and B

There are quadratic equation systems where a simple inspection learns us that one unknown can be eliminated without leaving any radical. In dimension three we fall immediately back in a two dimensional system which has an analytical solution. We provide two examples.

Case A. In the exceptional case where Φ_1 can be reduced to an equation of the shape $a\,w_i^2 + b\,w_j^2 = 0$ with $i \neq j$ and $1 \leq i, j \leq 3$ and the arbitrary non-zero constants a and b, we can for instance eliminate the co-ordinate w_i without the intervention of a radical in the two other equations. The latter two correspond to a system of two quadratic equations in two unknowns which can readily be solved and yield four solution vectors. If there are linear terms of w_i in this new reduced system there is duplication of the applicable system due to sign ambiguity of w_j. This then leads to eight solution-vectors.

Case B. If one unknown of the quadratic system only appears in a single shape either as square or in a particular bilinear term (by convention we have excluded systems with an unknown which only linearly appears in each equation) a linear combination of the equations can be performed such that this term disappears in two equations of the system. These two equations represent a two dimensional quadratic equation system with four analytic solution vectors. We stay with four solution vectors if the third unknown appeared in a bilinear combination, otherwise we get eight solution vectors altogether.

CASE III

Next we consider the quite important and common system which is void of linear unknowns and where $c_i \neq 0$ for at least one value of i. We thus start from three equations with the shape:

$$\Phi_i\,(H_i, \mathbf{0}, c_i) \,=\, 0 \tag{11}$$

implicitly with the unknown vector $\mathbf{w}$. For this case it is mandatory to ensure that H_1 is a positive definite matrix, if necessary (and feasible) obtained by means of linear combinations of the equations which simultaneously have to take care that, for instance, c_1 and c_2 are equal to zero. We further apply the necessary orthogonal transformation to all H_i in order to diagonalize H_1. The system is thereby modified into:

$$0 \,=\, \Phi_1(\Delta(e_1, e_2, e_3),\, \mathbf{0}, 0) \tag{12a}$$

$$0 \,=\, \Phi_2(F^{(2)},\, \mathbf{0}, 0) \tag{12b}$$

$$0 \,=\, \Phi_3(F^{(3)},\, \mathbf{0},\, c_3) \tag{12c}$$

where $0 < e_i$ for all i. To diagonalize Φ_2 we have to introduce a rescaling, such that (12a) simply becomes

$$0 \,=\, \Phi_1(\Delta(1,1,1),\, \mathbf{0}, 0) \,=\, u_1^2 + u_2^2 + u_3^2 \tag{13}$$

This is achieved by setting $u_i = w_i\sqrt{e_i}$. Thereby (13) becomes insensitive for a similarity transformation necessarily involving an orthogonal matrix. Let A_2 be the orthogonal matrix which is required in order to diagonalize (12b) by the transformation $\mathbf{v} = A_2\mathbf{u}$. Applying this also to (12c) yields a modified system which we can present as follows:

$$v_1^2 + v_2^2 = -v_3^2 \tag{14a}$$
$$\ell_1 v_1^2 + \ell_2 v_2^2 = -v_3^2 \tag{14b}$$
$$\mathbf{v}' \hat{H} \mathbf{v} = -c_3 \tag{14c}$$

From (14a) and (14b) we derive that $v_1^2 = m_1 v_3^2$ and $v_2^2 = m_2 v_3^2$, where m_1 and m_2 are known scalars. Substituting this in (14c) yields

$$\Lambda v_3^2 = -c_3 \tag{15}$$

with four values for Λ resulting from

$$\Lambda = |\pm\sqrt{m_1}, \pm\sqrt{m_2}, 1|\ \hat{H} \begin{vmatrix} \pm\sqrt{m_1} \\ \pm\sqrt{m_2} \\ 1 \end{vmatrix} \tag{16}$$

corresponding altogether to eight solution vectors $|v_1, v_2, v_3|'$. These vectors still have to undergo the inverse transformations to bring them in agreement with the original quadratic equations we had to solve at the start.

CASE IV

Looking back at (12) and (15), it is obvious that also the system

$$0 = \Phi_1(\Delta(e_1, e_2, e_3), \mathbf{0}, 0) \tag{17a}$$
$$0 = \Phi_2(F^{(2)}, \mathbf{0}, 0) \tag{17b}$$
$$0 = \Phi_3(F^{(3)}, \mathbf{b}, c_3) \tag{17c}$$

leads to an analytical solution. It means that we start from two equations where both the linear terms **and** the constants are zero. This must be so, because (17c) cannot be used to cancel the constants in the two previous equations contrary to what happened in case III. The equation (15) has now to be replaced by

$$\Lambda v_3^2 + 2(\pm b_1'\sqrt{m_1} + \pm b_2'\sqrt{m_2} + b_3')\, v_3 + c_3 = 0 \tag{18}$$

which also yields eight solution vectors. The accents are used to indicate that b_i' is the i-th component of $\mathbf{b}$ after the application of the linear transformations necessary to achieve the diagonalization of (17b).

5. SIMPLER NUMERICAL SOLUTIONS

By 'simpler' we mean those special cases which allow the elimination of two unknowns without the intervention of the analytical solution of a quartic, which may be required in the general case, in order to eliminate the second unknown. At any rate, the last step will be the numerical resolution of a higher degree polynomial of the unknown left after elimination.

CASE V

In case III we had a system without linear unknowns in all three equations. In this case we now allow such linear unknowns in one of the three equations, say for $i = 3$. Proceeding as in case III we readily obtain:

$$v_1^2 + v_2^2 = -\hat{c}_1 - v_3^2 \tag{19a}$$

$$\ell_1 v_1^2 + \ell_2 v_2^2 = -\hat{c}_2 - \ell_3 v_3^2 \tag{19b}$$

$$\mathbf{v}' \hat{H} \mathbf{v} + \mathbf{b_3} \mathbf{v} = -\hat{c}_3 \tag{19c}$$

where the carets indicate a major formal difference with case III. From the first two equations we obtain v_1 and v_2 as two radicals which are a function of v_3 and which we can write as follows:

$$v_1 = \pm\sqrt{R_1} = \pm\sqrt{a_1 + b_1 v_3^2}, \qquad v_2 = \pm\sqrt{R_2} = \pm\sqrt{a_2 + b_2 v_3^2} \tag{20}$$

Substituting these radicals in (19c) yields an equation of the form:

$$P_0 + P_1 \sqrt{R_1} + P_2 \sqrt{R_2} + P_3 \sqrt{R_1} \sqrt{R_2} = 0 \tag{21}$$

where P_0, P_1, P_2 and P_3 are known functions of v_3, and where the former is quadratic, P_1 and P_2 are linear and P_4 is a constant. The transformation of this expression to a rational function of v_3 yields an eighth degree polynomial. For every root of this polynomial one finds four acceptable combinations for the values of v_1 and v_2. This results into 32 solution vectors which have to be transformed back into the reference system of the original problem.

CASE VI

In this case we consider systems of three quadratic equations without bilinear combinations of the unknowns. By an adequate translation of the co-ordinate origin the linear terms of the first equation can be eliminated, and thus we are left with:

$$0 = \Phi_1(\Delta(e_{11}, e_{12}, e_{13}), \mathbf{0}, c_1) \tag{22a}$$

$$0 = \Phi_2(\Delta(e_{21}, e_{22}, e_{23}), \mathbf{b_2}, c_2) \tag{22b}$$

$$0 = \Phi_3(\Delta(e_{31}, e_{32}, e_{33}), \mathbf{b_3}, c_3) \tag{22c}$$

It is our aim to obtain equations which are only a function of two unknowns. Therefore we subtract equation (22a) multiplied by the adequate scalars once from (22b) and once from (22c) in order to cancel the coefficients of v_1^2 in (22b) and (22c). Next we subtract the new second equation multiplied by b'_{31}/b'_{32} from the the new third equation and the latter equation becomes:

$$e''_{32}\,v_2^2 \;+\; e''_{33}\,v_3^2 \;+\; 2\,b''_{32}\,v_2 \;+\; 2\,b''_{33}\,v_3 \;+\; c''_3 \;=\; 0 \tag{23}$$

where the accents denote successive modifications introduced by the subtractions. But (23) can be considered to be a quadratic equation of the unknown v_2 where the solution is a function of v_3. More specifically this means

$$v_2 \;=\; \frac{-\,b''_{32} \pm \sqrt{b''^2_{32} \;-\; e''_{32}\,q_y(v_3)}}{e''_{32}} \tag{24}$$

with the abbreviation

$$q_y(v_3) \;=\; e''_{33}\,v_3^2 \;+\; 2\,b''_{33}\,v_3 \;+\; c''_3$$

The same procedure can be used to eliminate the coefficients of v_2^2 and v_2 to determine the unknown v_1 as a function of v_3 as well. This leads to

$$v_1 \;=\; \frac{-\,b'''_{31} \pm \sqrt{b'''^2_{31} \;-\; e'''_{31}\,q_x(v_3)}}{e'''_{31}} \tag{25}$$

with the abbreviation

$$q_x(v_3) \;=\; e'''_{33}\,v_3^2 \;+\; 2\,b'''_{33}\,v_3 \;+\; c'''_3$$

Substituting (24) and (25) in (22a) and removing the radicals again yields a rational eight degree polynomial of v_3 which, due to the sign ambiguities in (24) and (25), results in 32 solution vectors.

Case VII

The missing squares in a quadratic equation system is an interesting case, because the Hermitian matrices involved will always remain indefinite independently of linear transformations. We will address this property in the last section. Consequently, the solution we propose is purely algebraic only involving a translation of the origin.

We start from a system which we can represent as follows:

$$D\begin{pmatrix} xy \\ xz \\ yz \end{pmatrix} \;+\; B\,\mathbf{x} \;=\; \mathbf{c} \tag{26}$$

where D and B are full rank 3×3 matrices. Multiplying (26) by D^{-1} reduces this equation system to:

$$xy + \mathbf{g_1'}\,\mathbf{x} = c_1' \tag{27a}$$

$$xz + \mathbf{g_2'}\,\mathbf{x} = c_2' \tag{27b}$$

$$yz + \mathbf{g_3'}\,\mathbf{x} = c_3' \tag{27c}$$

Case VII A Should one of the coefficients g_{13}, g_{22} or g_{31} be zero, then the corresponding equation system can be resolved **analytically**. Let us for instance assume that $g_{13} = 0$ then (27a) can be rewritten as

$$x = \frac{c_1' - g_{12}\,y}{y + g_{11}} \tag{28a}$$

Substituting this in (27b & c) yields equations of the following shape after removal of the nominator:

$$h_{22}'y^2 + 2\,h_{23}'yz + 2\,b_{22}'y + 2\,b_{23}'z = c_2'' \tag{28b}$$

$$y^2z + f_{22}'y^2 + 2\,f_{32}'yz + 2\,b_{32}'y + 2\,b_{33}'z = c_3'' \tag{28c}$$

Solving these equations for z yields:

$$z = -\frac{h_{22}'y^2 + 2b_{22}'y - c_2''}{2h_{23}'y + b_{23}'} \tag{29}$$

and

$$z = -\frac{f_{22}'y^2 + 2b_{32}'y - c_3''}{y^2 + 2f_{32}'y + 2b_{33}}$$

Equating the right hand sides of the two last equations and getting rid of the nominators results in a fourth degree polynomial of y. There are no ambiguities which could increase the number of the four solution vectors.

Case VII B. Should such an incidental simplification of case VII A not apply, we introduce the co-ordinate translation defined by:

$$u_1 = x + g_{12}, \qquad u_2 = y + g_{11}, \qquad u_3 = z + g_{21} \tag{30}$$

which simplifies (27a) to the shape

$$u_3 = \frac{c_1' - g_{11}'g_{12}' + g_{13}'g_{21}' - u_1\,u_2}{g_{13}'} = \alpha + \beta\,u_1\,u_2 \tag{31a}$$

while the two other equations become

$$u_1u_3 + g_{22}'u_2 + g_{23}'u_3 = c_2'' \tag{31b}$$

$$u_2u_3 + g_{31}'u_1 + g_{32}'u_2 + g_{33}'u_3 = c_3'' \tag{31c}$$

We now substitute u_3 of (31a) (31a) in (31b) and this leads to an equation of the following shape:

$$\begin{aligned} 0 &= (u_2)\, u_1^2 \,+\, 2\,(a_{20} \,+\, a_{22}u_2)\, u_1 \,+\, (b_{20} \,+\, b_{22}\, u_2) \\ &= (u_2)\, u_1^2 \,+\, 2\, K(u_2)\, u_1 \,+\, L(u_2) \end{aligned}$$

and this yields the value of u_1 as a function of u_2

$$u_1 \;=\; \frac{-K(u_2) \,\pm\, \sqrt{K^2(u_2) \,-\, u_2\, L(u_2)}}{u_2} \tag{32}$$

Substituting this result for u_1 back in (31a) provides with u_3 as a function of u_2, Substituting all this in (31c) yields a function of u_2 with the single radical appearing in (32) multiplied by a quadratic polynomial of u_2. Thus, in order that (31c) can be transformed into a rational polynomial of u_2 we have to put the following expression $\pm\,(b_{30} + b_{31}u_2 + b_{32}u_2^2)\sqrt{K^2(u_2) \,-\, u_2\, L(u_2)}$ in a separate equation member and square both members. In this way with get a six degree polynomial and thus six roots for u_2. The number six has to be multiplied by two due to the sign ambiguity in (32) when computing the complete solutions. Thereby we get twelve solution vectors.

CASE VIII

We draw the attention to the fact that the cases VI and VII have both involved three particular non-linear shapes in which all three unknowns were involved. We have checked that also the other possibilities can be solved very much the same way as in the previous cases. The cases meant are one square and two bilinear terms or two squares and one bilinear combination of the unknowns. The corresponding algorithms can be derived without problems.

6. Positive Definiteness and Sums of Matrices

In this section we recall some basics about the simplest rank deficient single symmetric matrices. Further, we study the ability to obtain a positive definite matrix starting from two full rank indefinite matrices. As before we will present this material limited to dimension three.

If H is a positive definite matrix, then $0 \,<\, \mathbf{x}'H\mathbf{x}$ holds for any arbitrary non-zero vector $\mathbf{x}$. Henceforth the scalar $\mathbf{u}'H\mathbf{v}$ will be represented by $H(\mathbf{u},\mathbf{v})$. If also equality to zero applies, we say that H is positive semidefinite. If $H(\mathbf{x},\mathbf{x})$ is positive for some vectors $\mathbf{x}$ and negative for others, we say that H is indefinite. When similar properties hold with the opposite inequality sign the matrix is negative (semi)definite. A necessary and sufficient condition for H to be positive definite requires that the determinant of all minors from dimension one to three centered on the diagonal of H are positive. For the minors of dimension one this means that all diagonal elements h_{11}, h_{22}, h_{33} are separately positive. For the two-dimensional minors for instance centered around $1, 1$ – that is with column one and row one suppressed – this requires that $0 < h_{22}h_{33} - h_{23}^2$ and similarly when suppressing the row and column two or three. The minor of dimension three is simply the determinant or $0 < \det(H)$. This is a still tractable procedure in dimension three but it becomes cumbersome if this has to be applied to sums of matrices, and that's our case.

For we usually require the normalized eigenvectors $\mathbf{u}_{i1}, \mathbf{u}_{i2}, \mathbf{u}_{i3}$ and the corresponding eigenvalues e_{i1}, e_{i2}, e_{i3} of the matrix H_i, we shall rather rely on these data which are subject to the following well known properties:

$$H_i \, \mathbf{u}_{ik} \;=\; e_{ik} \, \mathbf{u}_{ik} \tag{33a}$$

$$\mathbf{u}'_{ik} \, \mathbf{u}_{i\ell} \;=\; \delta_{k\ell} \qquad (\text{for } k,\, \ell \;=\; 1, 2, 3) \tag{33b}$$

where δ_{kl} is the Kronecker delta which is equal to one if $k = \ell$ and zero otherwise. The eigenvalues remain the same if (33a) is subjected to a similarity transformation by means of an orthogonal matrix A. This is obvious, because such a transformation reads

$$A \, H_i \, A'(A\mathbf{u}_{ik}) \;=\; e_{ik} \, (A\mathbf{u}_{ik})$$

From (33) one further derives that

$$H_i \;=\; \sum_{k}^{3} \; e_{ik} \, \mathbf{u}_{ik} \, \mathbf{u}'_{ik} \tag{34}$$

This result is known to be unique as far as the $\mathbf{u}_{ik}$ represent an orthonormal base and the corresponding eigenvalues are different. A breakdown similar to (34) exists for particular non-orthogonal bases, but, as far as we know, the analysis of such a breakdown has probably not yet been studied in detail. In contrast, the synthesis aspect is trivial, because for the non-zero m-dimensional vectors $\mathbf{p_k}$ with $1 \leq k \leq n$ where n may be any finite positive number, the matrix $M \;=\; \sum_{k=1}^{n} \mathbf{p_k} \, \mathbf{p'_k}$ is a $m \times m$ real symmetric matrix and thus Hermitian. It is this property which we employ to linearly combine two or three three-dimensional symmetric matrices to obtain a positive definite result, as far as feasible.

It is our aim to find values of μ_i for which

$$H_{tot} \;=\; \mu_1 \, H_1 \;+\; \mu_2 \, H_2 \;+\; \mu_3 \, H_3 \tag{35}$$

is positive definite. To avoid confusion with respect to mutual orthogonality of eigenvectors, we propose the following notation to start with:

$$H_1 \;=\; \sum_i \alpha_i \, \mathbf{u}_{ik} \, \mathbf{u}'_{ik}, \quad H_2 \;=\; \sum_i \beta_i \, \mathbf{v}_{ik} \, \mathbf{v}'_{ik}, \quad H_3 \;=\; \sum_i \gamma_i \, \mathbf{w}_{ik} \, \mathbf{w}'_{ik} \tag{36}$$

where the Greek characters represent eigenvalues, two of which may be zero for each matrix.

The simplest case we may encounter is then a combination of three symmetric matrices each of rank one. This means that each matrix has only one non-zero eigenvalue and (35) becomes

$$H_{tot} \;=\; \mu_1\,\alpha\,\mathbf{uu}' \;+\; \mu_2\,\beta\,\mathbf{vv}' \;+\; \mu_3\,\gamma\mathbf{ww}' \tag{37}$$

and whatever the sign of each eigenvalue may be, there will always be an infinite number of possibilities to obtain a positive definite matrix H_{tot}, provided the three eigenvectors involved are a (potentially non-orthogonal) base of the three dimensional Euclidean vector space, in other words, if we put them either in the columns or the rows of a 3×3 matrix, its determinant must be non-zero. Case VI was a case where the Hermitian matrices can be considered to derive from three rank-one matrices whose eigenvectors are mutually orthogonal. We can then always write

$$\begin{vmatrix} H_1 \\ H_2 \\ H_3 \end{vmatrix} = \begin{vmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \dots & & \\ \mu_{31} & \dots & \mu_{33} \end{vmatrix} \; \begin{vmatrix} \alpha\,\mathbf{uu}' \\ \beta\,\mathbf{vv}' \\ \gamma\mathbf{ww}' \end{vmatrix} \tag{38}$$

where each H_i will be full rank provided the matrix $M(\mu_{ij})$ is full rank and there is no $\mu_{ij} = 0$ for whatever combination of $1 \le (i, j) \le 3$. The inverse is subject to constraints which we do not address here.

The most basic symmetric matrix of rank two has the shape

$$\begin{vmatrix} 0 & h_{12} & 0 \\ h_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \frac{h_{12}}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} \begin{vmatrix} 1 & 1 & 0 \end{vmatrix} - \frac{h_{12}}{\sqrt{2}} \begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix} \begin{vmatrix} 1 & -1 & 0 \end{vmatrix} \tag{39}$$

as one derives from the characteristic equation $\lambda(\lambda^2 - h_{12}^2)$ whose roots for λ, here namely zero and $\pm h_{12}$, are the eigenvalues of the matrix. Symmetrically adding an arbitrary non-zero value h_{13} to the previous matrix still yields a characteristic equation with a zero root and the eigenvalues $\pm\sqrt{h_{12}^2 + h_{13}^2}$, thus still a rank two matrix. If we now add three such matrices where also h_{23} is non-zero, the characteristic equation has the shape

$$0 \;=\; \det \begin{vmatrix} -\lambda & p & q \\ p & -\lambda & r \\ q & r & -\lambda \end{vmatrix} = -\lambda^3 \;+\; \lambda\,(p^2 + q^2 + r^2) \;+\; 2\,p\,q\,r \tag{40}$$

This function of λ is a cubic parabola with two extrema. The locations of this maximum and minimum are obtained by simple differentiation and are at $\lambda \;=\; \pm\sqrt{(p^2 + q^2 + r^2)/3}$. Whatever the non-zero values of p, q and r are, there is thus at least one positive and one negative root/eigenvalue. Linear combinations of three matrices of rank two each with only a single non-zero h_{ij} can thus not become positive definite. This remains true if linear transformations are simultaneously applied to all three matrices.

Looking back at (35) we now start with only two full rank matrices H_i which are both indefinite. We look for a sum of these matrices which should be positive definite. If a matrix is negative definite it is of course not indefinite and changing the sign of the matrix makes it positive definite, thus a no-problem. If, however, a matrix has two negative eigenvalues. we just invert its sign and we get two positive eigenvalues. We may thus assume that we start with two matrices H_a and H_b with only a single negative eigenvalue each. In this particular case (35) and (36) can be replaced by

$$H_{tot} \;=\; \mu_a\, H_a \;+\; \mu_b\, H_b \tag{41}$$

and

$$H_a\, \mathbf{u}_a \;=\; -\, e_a\, \mathbf{u}_a, \qquad H_b\, \mathbf{u}_b \;=\; -\, e_b\, \mathbf{u}_b \tag{42}$$

where e_a and e_b are the absolute values of the two negative eigenvalues of the problem. The fact that we want to be sure that there are only two negative eigenvalues in (41) requires that both μ-values have to be positive.

If H_{tot} needs to be positive definite then also $0 \,<\, h_{tot}(\mathbf{u}_a, \mathbf{u}_a)$ and $0 \,<\, h_{tot}(\mathbf{u}_b, \mathbf{u}_b)$ must hold. This immediately translates into the inequalities

$$\begin{aligned} \mu_a\, e_a \;&<\; \mu_b\, H_b(\mathbf{u}_a, \mathbf{u}_a) \\ \mu_b\, e_b \;&<\; \mu_a\, H_a(\mathbf{u}_b, \mathbf{u}_b) \end{aligned} \tag{43}$$

But e_a, e_b, μ_a and μ_b are all positive by assumption. Hence, both $H_a(\mathbf{u}_b, \mathbf{u}_b)$ and $H_b(\mathbf{u}_a, \mathbf{u}_a)$ need to be positive. The necessary condition we found this way can thus be summarized by

$$0 < H_a(\mathbf{u}_b, \mathbf{u}_b), \qquad 0 < H_b(\mathbf{u}_a, \mathbf{u}_a) \tag{44a}$$

$$\frac{e_a}{H_b(\mathbf{u}_a, \mathbf{u}_a)} < \frac{\mu_b}{\mu_a} < \frac{H_a(\mathbf{u}_b, \mathbf{u}_b)}{e_b} \tag{44b}$$

We will hereafter demonstrate that these conditions are also sufficient.

Let us consider a general unit vector $\mathbf{u}$ which we decompose in a normalized but not necessarily orthogonal base $\{\mathbf{u}_a, \mathbf{u}_b, \mathbf{u}_c \,=\, (\mathbf{u}_a \times \mathbf{u}_b)/||\mathbf{u}_a \times \mathbf{u}_b||\}$. This implies that $\mathbf{u}_a$ and $\mathbf{u}_b$ are not collinear. Collinearity would anyway preclude finding an eligible ratio μ_b/μ_a able to make H_{tot} positive definite in (41). We thus require that $< \mathbf{u}_a, \mathbf{u}_b >= \cos\theta_0$ is smaller than one in absolute value. The decomposition of $\mathbf{u}$ is then

$$\mathbf{u} \;=\; \alpha\, \mathbf{u}_a \;+\; \beta\, \mathbf{u}_b \;+\; \gamma\, \mathbf{u}_c$$

The component along $\mathbf{u}_c$ only provides a favorable or positive contribution to $H_{tot}(\mathbf{u}, \mathbf{u})$, but even if this contribution would be missing, H_{tot} needs to remain positive definite without the help of this contribution. Consequently, we will set $\gamma = 0$ and consider

$$\mathbf{u} \;=\; \alpha\, \mathbf{u}_a \;+\; \beta\, \mathbf{u}_b, \quad \text{with} \quad \mathbf{u}^2 \;=\; 1 \;=\; \alpha^2 \;+\; \beta^2 \;+\; 2\,\alpha\,\beta\, \cos(\theta_0) \tag{45}$$

Translated in Euclidean geometry the vectors $\mathbf{u}, \alpha\mathbf{u}_a$ and $\beta\mathbf{u}_b$ fit into a triangle and consequently, $1 \leq |\alpha| + |\beta|$ and 'a fortiori' $1 \leq \alpha^2 + \beta^2$. Hence, $2\,\alpha\,\beta\,\cos(\theta_0) \leq 0$ where equality obviously holds if $\mathbf{u}_a \perp \mathbf{u}_b$. On these grounds we work $H_{tot}(\mathbf{u}, \mathbf{u})$ out in detail, namely:

$$H_{tot}(\mathbf{u}, \mathbf{u}) \;=\; (\alpha\,\mathbf{u}'_a \;+\; \beta\,\mathbf{u}'_b)\,(\mu_a\,H_a \;+\; \mu_b\,H_b)\,(\alpha\,\mathbf{u}_a \;+\; \beta\,\mathbf{u}_b) \;=$$
$$-\,(\alpha^2\mu_a e_a \;+\; \beta^2\mu_b e_b) \;+\; 2|\alpha\beta\cos(\theta_0)|\,(\mu_a e_a \;+\; \mu_b e_b)\}$$
$$+\,[\beta^2\,\mu_a H_a(\mathbf{u}_b, \mathbf{u}_b) \;+\; \alpha^2\,\mu_b H_b(\mathbf{u}_a, \mathbf{u}_a)] \tag{46}$$

If the inequalities of (43) are satisfied we can replace the last bracket of the right hand side of (46) by the smaller quantity $\alpha^2\mu_a e_a \;+\; \beta^2\mu_b e_b$. After the resulting simplifications (46) is modified into the inequality

$$+\,2|\alpha\beta\cos(\theta_0)|\,(\mu_a e_a \;+\; \mu_b e_b) \;<\; H_{tot}(\mathbf{u}, \mathbf{u})$$

which completes the proof of the claim that the inequalities (44) are not only necessary but also sufficient to guarantee the positive definiteness of H_{tot} as defined before.

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