NOTE 9A

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ANGULAR MOMENTA OF THE SOLAR SYSTEM

A. INVARIABILITY OF THE GLOBAL SYSTEM

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1. INTRODUCTION

When considering the **A**ngular **M**omentu**m** (AMm) of the **S**olar **SY**stem (SSY) we are used to limit this momentum to the mass point approximations of the sun and its planets. This is supported by the fact that the orbit of a planet around the sun corresponds to a constant AMm by Kepler's second law. Inertially, this viewpoint neglects the fact that the sun moves on a trajectory around the SSY's barycenter. Although this trajectory is usually referred to as an 'orbit', it is certainly not a Keplerian orbit, because the barycenter does mechanically not coincide with the location of a driving 'central force'. This neglect corresponds to an inertial AMm which, even if modest, has an impact on the individual bodies of the SSY. As we will show, also the sum of all inertial mass point **a**ngular **m**oment**a** (AMa) of sun and planets is theoretically not completely constant.

Hence, there is a contradiction with the Eulerian axiom which says that a body which is not subject to external torques or forces, has a constant AMm. We may assume that this applies as well to rigid bodies or rigid parts rotating independently or not and kept together by internal forces. The SSY is in fact such a system of masses subject to the internal gravitational forces, but there is no reason why these forces should lead to a variation of the inertial AMm of the SSY as a whole. This point of view is enforced by the observation that there is not the slightest evidence of an interference of such a specific AMm perturbation and celestial mechanics.

The only way out to satisfy Euler's axiom which we could identify, resides in a reaction of the SSY bodies counteracting the AMm variation linked to the barycentric offset. Alternatively, giving up Euler's invariance principle is not an option, because it gives rise to an other even more fundamental problem. If so, the mechanical energy of the SSY would not be constant even if all of its constituting bodies were rigid, as we will explain later. This note is devoted to the theoretical study of this subject and the deduction of a proposal which can resolve the problem just presented.

In the next section we start by analytically isolating the constant and variable parts of the AMm of the more important SSY masses taken as points. More important means including the sun and all planets. A succinct numerical analysis is presented in section 3.

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For the isolation of the actual variation we can look at two aspects. The first is the vector sum of the mass point inertial AMa and its change in size and orientation as a function of time. The second aspect is the evolution of the orientation of the SSY instantaneous inertia ellipsoid at each sampling point. Both aspects have to be considered together to make a statement about the possibility that the SSY as a whole would be subject to something like a constant precession.

In the fourth section we propose an analytical derived 'point mass – body coupling' not to mix up with the intuitive 'sun's orbit and sun spin coupling' proposed by Shirley J.H.(2006) and not subscribed by the present author. To find out whether the inherent conjecture in our theory is true or not, depends on the ability to overcome any contradiction with existing observations and even much more, to explain observations of phenomena observed in the SSY whose origin were unclear so far. Even without the conjecture, the theory presented leads to weak oscillatory *individual point mass AMa* whose frequencies are intimately linked to the synodic periods of all planet pair combinations. This is derived in the last section. When accepting the conjecture also the actual extended bodies are the subject of well defined corresponding **torq**ues (TRQs).

In the past three years the attention of the author went to the sun, but in that respect he was not able go beyond the result stating that there were no contradictions in the observations he had scrutinized. This is certainly not enough! More promising is the case of the Earth for which we could, so far, heuristically derive rather accurately two **C**handler **w**obble (CW) frequencies mentioned in the literature and corresponding to periods of approximately 435 days due to Jupiter's intervention and 411 days caused by Saturn independent of any measurement interpretations. The latter experimental period has been questioned by Kiryan D.G. and Kiryan G.V.(2012) based on the identification of inadequate measurement evaluations presented in the literature. Whether this is sufficient to claim the non-existence of the 411 day period is, in our opinion, still open.

At first sight, the true prove would consist in the reconstruction of the polar wobble path and minute amplitude variation of the spin vector on the basis of the proposed theory. Unfortunately, the accurate observation analysis of the CW reveals that there are phase jumps and jerks, see for instance Malkin Z. and Miller N.(2010) and references therein. These anomalies will hardly be explainable by the variable planetary N-body TRQ which is fully continuous. Moreover, the study performed by R.S. Gross(2000) says in its introduction: 'Evidence is presented that the CW is excited by a combination of atmospheric and oceanic processes, with oceanic bottom pressure being the dominant excitation mechanism'. With the N-body perturbation TRQs in the background we may still claim that, in the first instance, the TRQs are driving or initiating polar wandering, while atmospheric and oceanic processes constitute an interfering feedback. This may fit together. Nevertheless, the Earth is thus a complex case. Therefore, in the end, it looks as if the moon is probably the better test case, because it is an almost RIGID body allowing a reasonable verification of the conjecture proposed without too many intricacies. Lack of adequate pole wandering data of the moon may be the trouble in that case.

However, one has to notice, that it is extremely probable that the author will not have the survival time to complete this demanding undertaking in the light of the very high accuracy requirements, his isolation as pensioner and the lack of adequate ready-to-use infrastructure. He will have to leave the job to an interested person or team somewhere in the world wishing her/him or them 'good luck'.

By the way, this IS actually the purpose of MY notes.

The reading of this note is mandatory before going to the notes 9B dealing with the Earth (as far as it went and could reach the upload) and 9C giving our results for the sun. If the proposed theory is acceptable it could also be an element in the explanation of the Venusian winds, decadal meteorological events on Mars and the zonal dynamics of the observable gas atmospheres of the Jovian planets.

2 THE GLOBAL INERTIAL ANGULAR MOMENTUM

2.1 Basics

Let M_{sun} be the mass and $\mathbf{r_{sun}}$ the position vector of the sun with respect to the barycenter. Let M_i with $\mathbf{r_i}$ similarly be the masses and barycentric positions of the planets with the indices $i = 1$ to 8 counting from Mercury with $i = 1$ to Neptune with $i = 8$, respectively. The position of the barycenter may be taken to be inertially fixed and the position of the sun in this system is at all times constrained by

$$M_{sun}\,\mathbf{r_{sun}} \;=\; -\sum_{i=1}^{8} M_i\,\mathbf{r_i} \tag{1}$$

The units in tables and plots presented in this note will rely on the astronomical unit AU $= 149.598\,10^6$ km as length unit. Except if specified otherwise, the masses are measured in Earth masses (EM). For example: $M_{sun} = 332270.$ EM. For the unit of time we use the terrestrial solar 'day', if not explicitly replaced by 'yr'. We will call them 'plotting units'.

For completeness we remind the magnitude constraint applicable to $\mathbf{r_{sun}}$. We therefore represent the semi-major axis of planet i by a_i and eccentricity by e_i. Assuming that the four outer planets would all have their aphelia aligned and in the same direction away from the sun, the maximum distance $|\mathbf{r_{sun}}|$ from the barycenter is subject to the following constraint:

$$\sup|\mathbf{r_{sun}}| \;<\; \frac{1}{M_{sun}}\sum_{i=5}^{8} M_i\,a_i\,(1\,+\,e_i) \;\approx 1.579\;10^6\ \text{km} \tag{2}$$

where the value on the right hand side cannot be reached, because the apsidal lines of the different planetary orbits are not collinear. When comparing this with the solar radius ρ_s which is equal to 6.96 10^5 km, we find that $|\mathbf{r_{sun}}|$ must always remain within 2.3 ρ_s. The magnitude of $\mathbf{r_{sun}}$ can in fact also approach zero. This variable barycentric offset is at the origin of all what we will study in this note.

Let us now neglect the N-body interactions among the planets. Then the gravitational interactions of the planets with the sun are based on the fact that the sun is located at the origin as a common central force. This configuration corresponds by convention to a *heliocentric* reference system. The simple vectorial link between barycentric and heliocentric co-ordinates is depicted in Fig. 1 hereafter,

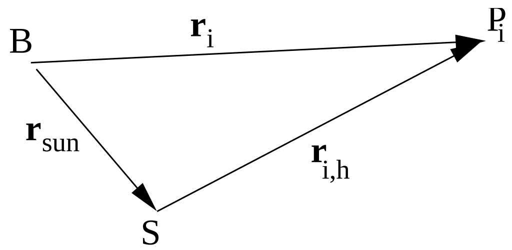

Figure 1. The vectors connecting the barycenter B, the sun S and planet Pi.

where the subscript h refers to the sun centered origin. Figure 1 translates into the simple vector sum:

$$\mathbf{r_{sun}} + \mathbf{r_{i,h}} = \mathbf{r_i} \tag{3}$$

Thus, the vector sum on the left is equivalent to the 'inertial' vector $\mathbf{r_i}$ and this also remains true if $\mathbf{r_{sun}}$ is expressed in terms of heliocentric vectors, because (1) can be rewritten as

$$\mathbf{0} = M_{sun} (\mathbf{0} + \mathbf{r_{sun}}) + \sum_{i=1}^{8} M_i (\mathbf{r_{i,h}} + \mathbf{r_{sun}}) \tag{4}$$

This further simplifies to

$$M_{tot} \, \mathbf{r_{sun}} = - \sum_{i=1}^{8} M_i \, \mathbf{r_{i,h}} \tag{5}$$

with the total SSY mass defined by

$$M_{tot} = M_{sun} + \sum_{i=1}^{8} M_i = 332716.7 \, EM \tag{6}$$

To express $\mathbf{r_{sun}}$ in terms of heliocentric planetary positions we will employ the dimensionless normalized masses:

$$m_{sun} = \frac{M_{sun}}{M_{tot}}, \qquad m_i = \frac{M_i}{M_{tot}} \tag{7}$$

It is further well known that the center of mass of a mass system coincides with the center of inertia of that system. The moment arm of the AMm of the sun as point mass thus corresponds to $\mathbf{r_{sun}}$ and its barycentric or inertial AMm is by definition equal to the following vector product:

$$\mathbf{D_s} = M_{sun} (\mathbf{r_{sun}} \times \dot{\mathbf{r}}_{\mathbf{sun}}) \tag{8}$$

We agree that the upper case D will by convention always refer to point masses. Thus, in the present context,

$$\mathbf{D_i} = M_i (\mathbf{r_i} \times \dot{\mathbf{r}}_{\mathbf{i}}) \tag{9}$$

represents the inertial AMm of planet i.

2.2 Splitting of the Global Inertial AMm

In this subsection we undertake the 'splitting' of the global inertial AMm. In fact this is the partitioning of the inertial AMm into a constant and a variable part. The constant part is identified by finding all vector products $\mathbf{r_{i,h}} \times \dot{\mathbf{r}}_{\mathbf{i,h}}$ for all planets, or $1 \leq i \leq 8$ in the heliocentric expression of the global inertial AMm including the sun. These particular vector products each multiplied by their constant planetary mass M_i correspond to the constant AMa of the different planetary orbits in agreement with Kepler's second law. This law applies to a very good approximation to all main planets of the SSY. The sum of all constant contributions yields then the reference direction with respect to which the variation of the global AMm is to be measured.

Let us start by deriving the inertial vector $\mathbf{D_s}$ in terms of the heliocentric planet position. Therefore we substitute (5) for $\mathbf{r_{sun}}$ and $\dot{\mathbf{r}}_{\mathbf{sun}}$ in (8). This yields:

$$\begin{aligned} \mathbf{D_s} &= M_{sun} \left(-\sum_{i=1}^{8} m_i \, \mathbf{r_{i,h}} \right) \times \left(-\sum_{j=1}^{8} m_j \, \dot{\mathbf{r}}_{\mathbf{j,h}} \right) \\ &= M_{sun} \sum_{i=1}^{8} \sum_{j=i+1}^{8} m_i \, m_j \, (\mathbf{r_{i,h}} \times \dot{\mathbf{r}}_{\mathbf{j,h}} + \mathbf{r_{j,h}} \times \dot{\mathbf{r}}_{\mathbf{i,h}}) + \\ & \quad M_{sun} \sum_{i=1}^{8} m_i^2 \, \mathbf{r_{i,h}} \times \dot{\mathbf{r}}_{\mathbf{i,h}} \end{aligned} \tag{10}$$

The very last term in (10) is obviously constant. It actually corresponds to the constant solar mass center AMm with the amplitude:

$$|\mathbf{D_{sc}}| = M_{sun} \, |\sum_{i=1}^{8} m_i^2 \, (\mathbf{r_{i,h}} \times \dot{\mathbf{r}}_{\mathbf{i,h}})| = 0.1349 \; 10^{-1} \; force \cdot length \tag{11}$$

Before proceeding, we introduce the abbreviation

$$\mathbf{p_{jk}} \;=\; m_j \, m_k \, (\mathbf{r_{j,h}} \times \dot{\mathbf{r}}_{\mathbf{k,h}} + \mathbf{r_{k,h}} \times \dot{\mathbf{r}}_{\mathbf{j,h}}) \tag{12}$$

for what we will call the mixed momentum of the bodies j and k, with $\mathbf{p_{jk}} = \mathbf{p_{kj}}$ and $\mathbf{p_{kk}} = m_k^2 \, \mathbf{r_{k,h}} \times \dot{\mathbf{r}}_{\mathbf{k,h}}$. The vector product $\mathbf{r_{sun}} \times \dot{\mathbf{r}}_{\mathbf{sun}}$ will reappear in the AMa of the planets and therefore we introduce the abbreviations $\mathbf{p_c} = \sum_{i=1}^{8} \; \mathbf{p_{ii}}$ and $\mathbf{p_v} = \sum_{i=1}^{8} \; \sum_{j=i+1}^{8} \mathbf{p_{ij}}$. For the planets the straight forward substitution of (3) and (5) into (9) leads to:

$$\begin{aligned} \mathbf{D_i} &= M_i \, (\mathbf{r_{i,h}} \times \dot{\mathbf{r}}_{\mathbf{i,h}} + \mathbf{p_v} + \mathbf{p_c}) + \\ & \quad M_i \, [\, \mathbf{r_{i,h}} \times \left(-\sum_{j=1}^{8} m_j \, \dot{\mathbf{r}}_{\mathbf{j,h}} \right) + \left(-\sum_{j=1}^{8} m_j \, \mathbf{r_{j,h}} \right) \times \dot{\mathbf{r}}_{\mathbf{i,h}} \,] \\ &= M_i \, (\mathbf{r_{i,h}} \times \dot{\mathbf{r}}_{\mathbf{i,h}}) - M_{tot} \sum_{j=1}^{8} \mathbf{p_{i,j}} + M_i \, (\mathbf{p_v} + \mathbf{p_c}) \end{aligned} \tag{13}$$

We are now in the position to explicitly write down the formulae for the global AMm $\mathbf{D_{ss}}$ of the SSY, namely

$$\mathbf{D_{ss}} \;=\; \sum_{i=1}^{8} M_i\,(\mathbf{r_{i,h}} \;\times \dot{\mathbf{r}}_{\mathbf{i,h}}) \;+\; M_{tot}\,(\mathbf{p_v} \;+\; \mathbf{p_c} \;-\; \sum_{j=1}^{8}\sum_{i=1}^{8} \mathbf{p_{ij}}) \tag{14}$$

But

$$\sum_{i=1}^{8}\sum_{j=1}^{8} \mathbf{p_{ij}} \;=\; 2\sum_{i=1}^{8}\;\sum_{j=i+1}^{8} \mathbf{p_{ij}} + \sum_{i=1}^{8} \mathbf{p_{ii}} \tag{15}$$

and consequently (14) simplifies to

$$\mathbf{D_{ss}} \;=\; \sum_{i=1}^{8} M_i\,(\mathbf{r_{i,h}} \;\times \dot{\mathbf{r}}_{\mathbf{i,h}}) \;-\; M_{tot}\,\mathbf{p_v} \tag{16}$$

From this we see that the mean AMm of the SSY is constant while the variable part

$$\mathbf{D_{tot,v}} \;=\; -\,M_{tot}\sum_{i=1}^{8}\;\sum_{j=i+1}^{8} \mathbf{p_{ij}} \tag{17}$$

oscillates around this mean, due to the nature of the pseudo periodic variation as a function of time of the different $\mathbf{p_{ij}}$ vectors defined by (12). The proposal worked out in section 4 for the SSY body AMm contributions cancels the variable angular momentum $\mathbf{D_{tot,v}}$.

3 NUMERICAL ASPECTS

3.1 Accuracy of Ephemerides Used

Whatever size the variable magnitude ratio $\mathbf{D_{tot,v}}/\mathbf{D_{tot,c}}$ may be, $\mathbf{D_{tot,v}}$ remains a perturbation, if not canceled in one way or another. At SSY level such a cancellation mechanism is hardly conceivable, leaving us with the potential alternative discussed in next section.

The question we face nevertheless, is:"are the ephemerides we use accurate enough for whatever alternative?". These orbital data are, in fact, gained from the analytical epheme rides of Van Flandern T.C. and Pulkkinen K.F.(1979) which claim to provide heliocentric position vectors accurate to approximately 0.02^0. Nothing is said about distance magnitudes and continuity. When, in the second half of 2010, the author started the study about the variable AMm of the sun, he had these ephemerides already programmed and considered them to be sufficient. Looking at the plots of note 9C, this viewpoint appears to be satisfactory. Especially, the continuity aspect allowed the conspicuously flawless computation of two successive partial time derivatives. To this aim we implemented a five point based numerical differentiation (see Stiefel (1963)) with a time step of 10 days for Jupiter, 20 days for Saturn, 40 days for Uranus and 80 days for Neptune. This thus corresponds, for example, to an orbital arc covering forty days for Jupiter and 320 days for Neptune to compute derivatives for a single epoch.

When, in 2013, we looked at the minute AMm perturbations hidden in the CW of the Earth, it was immediately obvious that the inaccuracies of the analytical ephemerides exceed by far the effects we wished to analyze. In a full time job one certainly would acquire in some months the adequate NASA planetary and terrestrial ephemerides with the right characteristics, insert them in the system, etc. Only first then the actual integration study with the Euler differential equations can start. As a consequence, the author does not see enough usable time left in front of himself anymore. It's over.

Limited to the Jovian planets, we have nevertheless casted a glance at the numerical quality of the analytical heliocentric ephemerides at hand. To that aim we have computed the mean vector product $\mathbf{r_{i,h}} \times \dot{\mathbf{r}}_{\mathbf{i,h}}$ magnitudes – which should be constant – and their orientations based on three samples a year starting from 1800 up to the year 2000. Hereafter, we provide an edited printout of our corresponding analysis program. In this listing we

NUMERICAL QUALITY OF HELIOCENTRIC MOMENTA

JUPITER Mn($\mathrm{h_5}$)=0.392275835D-01, RMSQ($\mathrm{h_5}$)=0.392275840D-01, σ=0.2515D-06
JUPITER $\mathbf{h_5}/||\mathbf{h_5}||$: 0.225418797D-01, 0.374058679D-02, 0.999738870D+00
JUPITER Mn $\alpha = 9.421776^0$, $\lambda = 1.309330^0$, Mn $\varepsilon = 0.0126^0$, RMSQ $\varepsilon = 0.0144^0$
JUPITER Mn($\dot{\mathrm{h}}_5$)=0.132915340D-07, RMSQ($\dot{\mathrm{h}}_5$)=0.157199145D-07, σ=0.2428D-08

SATURN Mn($\mathrm{h_6}$)=0.531578361D-01, RMSQ($\mathrm{h_6}$)=0.531578797D-01, σ=0.2773D-05
SATURN $\mathbf{h_6}/||\mathbf{h_6}||$: 0.400370092D-01, 0.168367844D-01, 0.999056264D+00
SATURN Mn $\alpha = 22.80823^0$, $\lambda = 2.489320^0$, Mn $\varepsilon = 0.0187^0$, RMSQ $\varepsilon = 0.0216^0$
SATURN Mn($\dot{\mathrm{h}}_6$)=0.495741832D-07, RMSQ($\dot{\mathrm{h}}_6$)=0.614125564D-07, σ=0.1184D-07

URANUS Mn($\mathrm{h_7}$)=0.755772934D-01, $RMSQ(h_7)$=0.755773825D-01, σ=0.4726D-05
URANUS $\mathbf{h_7}/||\mathbf{h_7}||$: 0.129244447D-01, -0.382917760D-02, 0.999909139D+00
URANUS Mn $\alpha = -16.50319^0$, $\lambda = 0.77236^0$, Mn $\varepsilon = 0.0050^0$, RMSQ $\varepsilon = 0.0056^0$
URANUS Mn($\dot{\mathrm{h}}_7$)=0.125248381D-06, RMSQ($\dot{\mathrm{h}}_7$)=0.139393900D-06, σ=0.1415D-07

NEPTUNE Mn($\mathrm{h_8}$)=0.950265232D-01, RMSQ($\mathrm{h_8}$)=0.950266720D-01, σ=0.6850D-05
NEPTUNE $\mathbf{h_8}/||\mathbf{h_8}||$: 0.234901242D-01, 0.201790029D-01, 0.999520328D+00
NEPTUNE Mn $\alpha = 40.66396^0$, $\lambda = 1.77458^0$, Mn $\varepsilon = 0.0185^0$, RMSQ $\varepsilon = 0.0211^0$
NEPTUNE Mn($\dot{\mathrm{h}}_8$)=0.200459017D-06, RMSQ($\dot{\mathrm{h}}_8$)= 0.222806114D-06, σ=0.2235D-07

employed the shorthands Mn = mean value, $\mathbf{h_i}$ = heliocentric angular momentum of planet i and $\mathrm{h_i}$ for its magnitude, σ = variance. α and λ are the mean right ascension and colatitude of the angular momenta computed with the method given in appendix B. The mean deviations with respect to this mean directions is denoted by ε. These deviations are obtained in turn by taking the arc cosine of the projection of the individual directions onto the mean. From all this we may conclude that the vector product directions are really accurate up to approximately 0.02^0 notwithstanding the intervention of the differentiations.

The fourth line for each separate planet contains the magnitudes of the differentiated AMa which should theoretically be zero. These magnitudes are in all cases smaller than the variances of the AMm in the first line. This is satisfactory for the aims we pursue. Nevertheless, the variances of the AMa indicate that, except for Jupiter, the AMa have hardly more than 4 reliable significant digits to offer. However, this is more than sufficient for all detailed plotting of theoretical curves.

3.2 Inertia Ellipsoid of the Solar System

As announced in the introduction, although lacking any valid theory, there is an interest for analyzing the time evolution of the inertia tensor of the complete SSY, thus comprising the planets and the sun. At any rate, the inertia tensor $\mathbf{N}$ of the SSY is not constant. We are especially interested to see how the little motion of the largest principal moment of inertia varies with respect to the direction of the global constant planetary AMm of the SSY.

To derive the components n_{ij} of the 3×3 symmetric matrix $\mathbf{N}$ we employ its conventional definition in the case of discretely distributed point masses M_i with their inertial barycentric positions $\mathbf{r_i}(x_i, y_i, z_i)$ where $1 \leq i \leq 8$ stands for the planets and $i = 9$ corresponds to the sun. We then have:

$$n_{xx} = \sum_{i=1}^{9} M_i(y_i^2 + z_i^2), \qquad n_{yy} = \sum_{i=1}^{9} M_i(x_i^2 + z_i^2), \qquad n_{zz} = \sum_{i=1}^{9} M_i(x_i^2 + y_i^2), \tag{18}$$
$$n_{xy} = \sum_{i=1}^{9} M_i(x_i\, y_i) = n_{yx}, \quad n_{xz} = \sum_{i=1}^{9} M_i(x_i\, z_i) = n_{zx}, \quad n_{yz} = \sum_{i=1}^{9} M_i(y_i\, z_i) = n_{zy}.$$

The three eigen-values of this symmetric matrix correspond to the magnitude of the three major axes of principal inertia along the corresponding mutually perpendicular eigen-vectors in ecliptic co-ordinates. This allows us to compute the instantaneous volume and the motion of largest principal moment of the inertia ellipsoid with respect to the direction of the sum of the planetary heliocentric AMa.

Added at the end of this note and based on three samples a year, figure 2 shows the evolution of the volume of the ellipsoid in the last two centuries. The volume curve indicates that it corresponds to a non-stochastic discrete spectrum which seems to be a sum of the first harmonics of the orbital periods of the Jovian planets. This would not be unexpected, because the diagonalized inertia tensor is filled with squared co-odinates of the different planetary positions. These coordinates correspond to squares of sines and cosines involving the simple or fundamental orbital periods. Especially, the actual periodicities could uncover some unexpected influences of mutual planetary configurations which are not obvious enough in an other context, even if the present considerations do not provide any explanation for a possible causality.

The dotted curve added at the bottom of figure 2 shows the colatitude evolution of the largest principal moment of inertia with respect to the constant part of the SSY AMm. The deeper minima of the volume values are accompanied by large colatitudes. Both curves are visibly related, but the colatitude evolution is not smooth. We compare this further with the right ascension of the largest principal axis with respect to the global fixed AMm displayed in figure 3 covering 4 centuries. Also the right ascension is not performing regular clean rotation(s) around the constant SSY AMm, although it also shows repeating configuration intervals over time. Consequently, as can be inferred from this little analysis, there is nothing that suggests a (pseudo) precession. It seems rather to be the contrary, the planetary orbits have the lead and not the AMa.

3.3 Analysis of $\mathbf{D_{tot,v}}$

The variable part of the SSY AMm, namely $\mathbf{D_{tot,v}}$ – applicable if the conjecture proposed in the next section is to be rejected – is equal to $(-M_{tot}\sum_{i=1}^{8}\sum_{j=i+1}^{8}\mathbf{p_{ij}})$. But the variable AMm of the sun is equal to $(+M_{sun}\sum_{i=1}^{8}\sum_{j=i+1}^{8}\mathbf{p_{ij}})$ in (10). The latter is presented in (5) in note9C and shown in its plots (1) and (2) with a different sign and a mass coefficient M_{sun} instead of M_{tot}. The plots (1) and (2) of note9C are thus sufficient to get a feeling for the variation of AMa and TRQs.

4. POINT MASS AND BODY ANGULAR MOMENTUM COUPLING

In this section we assume that on top of the constancy of the velocity of the center of mass, also the AMm of the SSY in inertial space has to remain constant, based on a generalization of an axiom formulated by Euler(1775 or 1967) and which is applicable to a freely moving rigid body not subject to TRQs or external forces. Therefore, we explain how Euler's principle of constant AMm has to be interpreted for systems of $2 < N$ extended bodies whose separate inertial motions are known. These motions must be a consequence of forces inside the system, while no forces or TRQs external to the system may be involved. We further assume that these bodies are coherent or, in other words, do not join or leave the system and keep their separate masses constant, but, on the other hand, they do not need to be rigid.

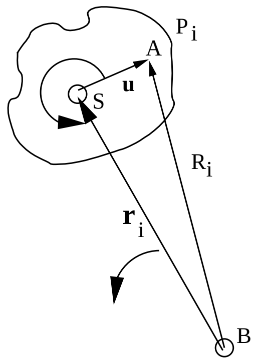

FIGURE 4. The body Pi, itself rotating around its instantaneous center of mass S and further moving around the barycenter B

We start with looking which types of AMa exist for the separate system bodies. To this end we consider Fig. 4 with the arbitrary i-th extended rigid or non-rigid and spherical or non-spherical body $\mathrm{P_i}$ of the system. Its instantaneous mass center is located at S. This mass center S is specific for the AMm of the body $\mathrm{P_i}$ rotating in isolation. Further, the body as a whole is assumed to move (around or) with respect to the inertial barycenter B.

To be able to represent the total 'system' AMm of $\mathrm{P_i}$, we agree to say that dm is a mass element at the point A located at $\mathbf{u}$ from S inside $\mathrm{P_i}$ with $\int_{\mathbf{P_i}} dm = M_i$. The vectors $\mathbf{u}$ and their time derivatives are measured in a reference system with origin S and with co-ordinate axes which are parallel to those used to describe the motion around B. The motion of S around B and its time derivative is described by $\mathbf{r_i}$ and $\mathbf{\dot{r}_i}$, respectively.

Then the total instantaneous inertial AMm $\mathbf{H_i}$ of the body with index i is known to be equal to:

$$\begin{aligned} \mathbf{H_i} &= \int_{\mathbf{P_i}} \mathbf{R_i} \times \mathbf{\dot{R}_i}\, dm = \int_{\mathbf{P_i}} (\mathbf{r_i} + \mathbf{u}) \times (\mathbf{\dot{r}_i} + \mathbf{\dot{u}})\, dm \\ &= M_i\,(\mathbf{r_i} \times \mathbf{\dot{r}_i}) + \int_{\mathbf{P_i}} \mathbf{u} \times \mathbf{\dot{u}}\, dm + \int_{\mathbf{P_i}} (\mathbf{r_i} \times \mathbf{\dot{u}} + \mathbf{u} \times \mathbf{\dot{r}_i})\, dm \end{aligned} \tag{18}$$

Hence, the value of $\mathbf{H_i}$ can be broken down in three parts, namely:

$$\mathbf{D_i} = M_i\,(\mathbf{r_i} \times \mathbf{\dot{r}_i}), \qquad \mathbf{B_i} = \int_{\mathbf{P_i}} \mathbf{u} \times \mathbf{\dot{u}}\, dm, \qquad \mathbf{A_i} = \int_{\mathbf{P_i}} (\mathbf{r_i} \times \mathbf{\dot{u}} + \mathbf{u} \times \mathbf{\dot{r}_i})\, dm \tag{19}$$

The barycentric AMm $\mathbf{D_i}$ is the equivalent 'mass point trajectory' contribution, the vector $\mathbf{B_i}$ is the 'i-th body' attitude AMm and $\mathbf{A_i}$ is a potential 'body asymmetry' contribution. The latter will cancel if for the mass element at any arbitrary point $\mathbf{u_0}$ inside $\mathrm{P_i}$, there is an equal mass element at $-\mathbf{u_0}$, in other words: if the center of mass of $\mathrm{P_i}$ is a point of mass distribution symmetry. This thus also applies, for example, to ellipsoidal shapes and layered mass distribution densities.

If there is central asymmetry, $\mathbf{A_i}$ would be absorbed in a potentially time dependent inertia tensor applicable to $\mathrm{P_i}$, which we have exercised in previous section. The problem then is the inability to define a meaningful spin vector. However, if $\mathrm{P_i}$ happens to be rigid, its inertia tensor $\mathbf{N_i}$ is constant and the conventional representation of the body AMm applies, namely:

$$\mathbf{N_i}\,\mathbf{\Omega_i} = \mathbf{B_i} + \mathbf{A_i} \tag{20}$$

with the spin velocity vector $\mathbf{\Omega_i}$. Thus, also in that case asymmetry can be ignored.

Let us now consider the total AMm $\mathbf{H_{sys}}$ of a barycentric system of N bodies with the angular momenta $\mathbf{D_i}$, $\mathbf{B_i}$ and $\mathbf{A_i}$, then we can write:

$$\mathbf{H_{sys}} = \sum_{i=1}^{N} \mathbf{H_i} = \sum_{i=1}^{N} (\mathbf{D_i} + \mathbf{B_i} + \mathbf{A_i}) \tag{21}$$

In this equation the vectors $\mathbf{D_i}$ are based on ephemerides or, in other words, the orbital trajectories one has obtained by integrating the differential equations of orbital motion taking all meaningful perturbations into account. In this way the coupling of orbital dynamics with AMa has been accomplished, also for the part going beyond Kepler's second law. At any rate, as mentioned before, mutual orbital perturbations of planets are quite small in a time span of centuries, see Brouwer D. and van Woerkom A.J.J.(1950). The actual reason why the AMa $\mathbf{D_i}$ of the planets are not constant is the fact that these $\mathbf{D_i}$ are the angular momenta around the barycenter and not the sun center!

If the N-body system is not subject to external TRQs or forces, the global AMm has to be constant which implies $\dot{\mathbf{H}}_{\mathbf{sys}} = \mathbf{0}$. Hence,

$$\dot{\mathbf{D}}_{\mathbf{i}} \;=\; -\,(\dot{\mathbf{B}}_{\mathbf{i}} \,+\, \dot{\mathbf{A}}_{\mathbf{i}}) \qquad (\text{for } i = 1, \ldots, N) \tag{22}$$

for all bodies separately, because $\mathbf{H_{sys}}$ has to be constant for whatever trajectories may have led to the values of $\mathbf{D_i}$. The integration of (22) brings us back to the AMa, but now with the constant parts stripped off. The proposed axiom extension represented by (22) can obviously not claim that the global AMm is constant if the system is not conservative, but it nevertheless states that the time derivative of the global TRQ is zero. Hence, the global isolated system cannot reorient itself inertially if this only implies internal TRQs.

This can be complemented with an important remark about energy. Assume that a N-body planetary system consists of rigid bodies only. In that case we expect conservation of energy. All bodies involved now satisfy (20). For any given body i we assume that $\mathbf{\Omega_i}$ can be divided up in a constant part $\mathbf{\Omega_{i0}}$ and a variable part $\mathbf{\Omega_{iv}}$. Then (22) says that $\dot{\mathbf{B}}_{\mathbf{i}} \;=\; -\,\mathbf{N_i}\,\dot{\mathbf{\Omega}}_{\mathbf{iv}}$. Moreover, we are sure that the mass point TRQ $\dot{\mathbf{B}}_{\mathbf{i}} \neq \mathbf{0}$ except at zero crossings (occurrences with theoretical measure zero). Non-zero TRQs impart energy to the bodies involved and if this energy is not compensated by opposite torques the system is not conservative. A not acceptable situation. The only way out we see, is to accept (22). This means in this case that mass point energy varies in a way opposite to the body energy, namely:

$$\mathbf{B_i} \;=\; -\,\mathbf{N_i}\,\mathbf{\Omega_{iv}}, \qquad \mathbf{\Omega_{iv}}\dot{\mathbf{B}}_{\mathbf{i}} \;=\; -\,\mathbf{\Omega_{iv}}\,\mathbf{N_i}\,\dot{\mathbf{\Omega}}_{\mathbf{iv}} \tag{23}$$

It is important to realize that (22) is not an exchange of AMm. Equation (22) is just stating that a change of barycentric mass point AMm of a body i is not possible without an equal and opposite change of the attitude AMm of the corresponding body (action and reaction), provided the asymmetry AMm $\mathbf{A_i}$ can be neglected or absorbed in a constant inertia tensor.

5. PERIODICITIES

We have seen in (17) that the mixed momenta $\mathbf{p_{jk}} \;=\; m_j\, m_k\, (\mathbf{r_{j,h}} \times \dot{\mathbf{r}}_{\mathbf{k,h}} \,+\, \mathbf{r_{k,h}} \times \dot{\mathbf{r}}_{\mathbf{j,h}})$, already introduced in (12), are the only variable contributors to $\mathbf{D_s}$ and $\mathbf{D_i}$ in (10) and (13), respectively. Consequently, this variability is dependent on the orbital periods T_k and T_j of the planets k and j, or equivalently the frequencies $\omega_i = 2\pi/T_k$ and $\omega_j = 2\pi/T_j$, respectively. Finding the fundamental frequencies in the Fourier series representing the ecliptic z-components of the mixed vector products in $\mathbf{p_{kj}}$ implies the consideration of circular orbits. This corresponds to set $x_k = a_k\, \cos{(\omega_k\, t \,+\, \phi_i)}$ and $y_k = a_k\, \sin{(\omega_k\, t \,+\, \phi_i)}$ for the projection of the motion of planet k onto the ecliptic, where the phase angle ϕ_k applies at a given arbitrary time $t = 0$. From a basic frequency point of view one verifies that

$$\begin{aligned}(\mathbf{p_{kj}})_z &\sim (x_k\, \dot{y}_j \,-\, y_k\, \dot{x}_j) \,+\, (x_j\, \dot{y}_k \,-\, y_j\, \dot{x}_k) \\ &= a_k\, a_j\, (\omega_k \,+\, \omega_j)\, \cos{[(\omega_k \,-\, \omega_j)t \,+\, \phi_k \,-\, \phi_j]}\end{aligned} \tag{24}$$

This yields the fundamental period $T_{kj} = |T_k^{-1} - T_j^{-1}|^{-1}$ of that term in the mixed momentum brought about by the mixed cross products of planet k and j. They are the synodic periods involving the planet pairs and correspond each to the time separating two successive conjunctions or oppositions. Because all planets rotate in the same direction around the sun, a synodic period is always larger than the smallest orbital period of the planet pair concerned. A closer approximation of the actually elliptical orbits is obtained by the higher harmonics of the synodic frequencies. The period of the first harmonic is half the period of the fundamental frequency.

The variable TRQ corresponding to AMm is by definition equal to $\dot{\mathbf{D}}_{\mathbf{i}}$ and $\dot{\mathbf{D}}_{\mathbf{s}}$. Hence,

$$\begin{aligned}(\dot{\mathbf{p}}_{\mathbf{kj}})_z &\sim (x_k\,\ddot{y}_j - y_k\,\ddot{x}_j) + (x_j\,\ddot{y}_k - y_j\,\ddot{x}_k) \\ &= -a_k\,a_j\,(\omega_k^2 - \omega_j^2)\,\sin[(\omega_k - \omega_j)t + \phi_k - \phi_j] \end{aligned} \tag{25}$$

and the TRQs are subject to the same synodic periods already found for the AMa.

Looking back at the different mixed moments occurring in $\mathbf{D_k}$ for planet k, we observe that the larger contributions come from

$$\mathbf{D_{vk}} \approx M_k\,(\sum_{i=1}^{k-1} + \sum_{i=k+1}^{8})\mathbf{p_{ik}} \tag{26}$$

From this we derive that – if there is some measurable effect of the AMa and TRQs on planet k – we may expect that the planets Jupiter and Saturn contribute most. Consequently, the synodic periods containing the orbital period of these two planets may be involved in these measurable effects. Therefore, we have listed hereafter these periods expressed in years, excluding Mercury.
$T_{25} = 0.6486$, $T_{35} = 1.0921$, $T_{45} = 2.2355$, $T_{65} = 19.859$, $T_{75} = 13.812$, $T_{85} = 12.782$,
$T_{26} = 0.6289$, $T_{36} = 1.0351$, $T_{46} = 2.0093$, $T_{56} = 19.859$, $T_{67} = 45.364$, $T_{68} = 35.871$.

Especially for a terrestrial (Earth like) planets i, we may consider the interplay of two different Jovian planets with the orbital period of that single terrestrial planet. In this context we make the following simple consideration. Let $T_i < T_j < T_k$, then $T_{ij} < T_{ik}$. On that basis we may look for the period $T_{ij/ik}$ of the separation of an ij conjunction from an ik conjunction. This is given by:

$$\frac{1}{T_{ij/ik}} = \frac{1}{T_{ij}} - \frac{1}{T_{ik}} = \frac{1}{T_{jk}} \tag{27}$$

This means that the synodic period T_{56}, for instance, has the potential to be detectable in the disturbance AMa on all terrestrial planets.

When considering the variable AMm of the sun in (10) it is obvious that the largest contributions come from the Jovian planets and periodicities of the mixed moments can first be considered pair wise. But we can also take the four Jovian planets in combined conjunctions in the same way as we did in (27). Thereby we obtain $T_{57/68} = T_{56/78} =$ 22.46 yr which corresponds exactly to the period of the mean Hale cycle of solar activity, see P.R. Wilson(1994). This is probably a physically meaningful coincidence, supporting the assumption that the variable AMa of the Jovian planets play a role in solar activity.

It now happens that the ratios of any two of the synodic periods only involving Jovian planets are very close to the ratio of two integers. This was already mentioned by Ariaga(1955) for Jupiter and Saturn, while Jose(1965) extended the list without being complete. These integer ratios explain why these periods have an approximate Smallest Common Multiple Period (SCMP) of quite modest size. We propose to approximate this SCMP by $178.75\,\mathrm{yr} = 9.0009\,T_{56}$, number also mentioned by Jose(1965) in the context of periodicities he analyzed. We then further note that 178.75 corresponds to 13 (12.942) times T_{57}, to 14 (13.985) times T_{58}, to 4 (3.9403) times T_{67}, to 5 (4.9831) times T_{68}, and to once (1.0428) T_{78}. Hence, we may really expect some periodicities very close to this SCMP in all relevant theories involving the synodic periods of the exterior planets.

Important is the fact that the SCMP is also at the origin of a very specific and similar quasi periodic planetary configuration. This configuration gives rise to a periodically reappearing three-foil shaped projection onto the ecliptic of the solar trajectory around the barycenter. The reappearance of this well known phenomenon every 178.75 yr has been studied for its long periodic recurrence by numerous solar physicists, see Charvatova I. and Střeštík J.(1991) and Charvatova(2000) and references therein. Apparent similarities of the corresponding AMa linked to the three-foils, slowly drift in their details or simply dissappear over time under the influence of the imperfection of the common multiple. As shown before, all synodic periods containing Uranus contribute most to this imperfection. This drift of similarities might be in line with the observations made in the paper by Abreu et al.(2012) concerning longer periods up to millenaries with typical periodicities in solar activity which disappear for a longer time and reappear later again.

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APPENDIX I

The parameters shown hereafter are extracted from the pocket atlas by J. Hermann(2000).

Orbital Parameters of the Major Planets

	a(AU)	e	M(EM)	Sidereal Period(yr)	Inclination (deg)
Mercury	0.387	0.206	0.055	0.241	$7^0\,0.3'$
Venus	0.723	0.007	0.815	0.615	$3^0\,23.7'$
Earth	1.000	0.017	1.000	1.000	–
Mars	1.524	0.093	1.07	1.881	$1^0\,51.0'$
Jupiter	5.203	0.048	317.89	11.862	$1^0\,18.3'$
Saturn	9.539	0.055	95.18	29.458	$2^0\,29.3'$
Uranus	19.191	0.047	14.54	84.015	$0^0\,46.3'$
Neptune	30.061	0.010	17.13	164.79	$1^0\,46.3'$

APPENDIX II

Determining the Mass Center of Image
of a Coherent Cloud of Points on a Hemisphere

By a small cloud we mean a set of $3 << N$ points p_i which are confined in an arbitrary small circle whose arc radius is less than - , or equal to $\pi/2$ on the unit sphere. The latter condition means that the complementary Cartesian reference with the co-ordinates x_i, y_i, z_i for the p_i points - or directions - can be rotated in such a way that this small circle is contained inside a hemisphere bounded by the equator containing the x and y axes, for instance.

By 'mass center of image' we mean the point C which is closest in the mean to all points of the cloud. By choosing polar co-ordinates having their pole on the z-axis, we can write $x_c = \cos\alpha\sin\lambda$, $y_c = \sin\alpha\sin\lambda$ and $z_c = \cos\lambda$, where α is the right ascension and λ the colatitude of C. The center of image, or $C(\alpha, \lambda)$, can be found by minimizing the following cost function

$$Q(\alpha,\lambda) \;=\; \sum_i^N (x_i \,-\, \cos\alpha\sin\lambda)^2 \,+\, \sum_i^N (y_i \,-\, \sin\alpha\sin\lambda)^2 \,+\, \sum_i^N (z_i \,-\, \cos\lambda)^2 \qquad (A)$$

with respect to α and λ. We further agree to write $\hat{x}, \hat{y}, \hat{z}$ for $\sum_i x_i$, $\sum_i y_i$, $\sum_i z_i$, respectively and compute the partial derivative of Q with respect to α. This derivative must be zero to reach the minimum of Q with respect to α. We get

$$\frac{\partial Q}{\partial \alpha} \;=\; \sin\lambda\,(\hat{x}\,\sin\alpha \,-\, \hat{y}\,\cos\alpha) \;=\; 0 \qquad (B)$$

which simply yields

$$\tan\alpha = \hat{y}/\hat{x} \qquad (C)$$

To resolve the ambiguity concerning the quadrant applicable to α one just looks at the actual signs of $\hat{x}$ and $\hat{y}$ which define the quadrant or employ the $\arg(\hat{x}, \hat{y})$ function from basic complex function theory, or the intrinsic function (D)ATAN2 in FORTRAN, which also does this job for us.

The value of λ is derived from the condition

$$\begin{aligned} \frac{\partial Q}{\partial \lambda} \;&=\; 0 \\ &=\; \cos\alpha\,\cos\lambda\,(\hat{x} \,-\, N\,\cos\alpha\sin\lambda) \\ &\quad\; +\, \sin\alpha\,\cos\lambda\,(\hat{y} \,-\, N\,\sin\alpha\sin\lambda) \,-\, \sin\lambda\,(\hat{z} - N\,\cos\lambda) \qquad (D) \end{aligned}$$

which simplifies to

$$\tan\lambda \;=\; \frac{\hat{x}\,\cos\alpha + \hat{y}\,\sin\alpha}{\hat{z}} \qquad (E)$$

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FIGURE 2. Volume of Inertia Ellipsoid and
Colatitude of Largest Principal Moment of
Inertia w.r.t. SSY AMm Direction

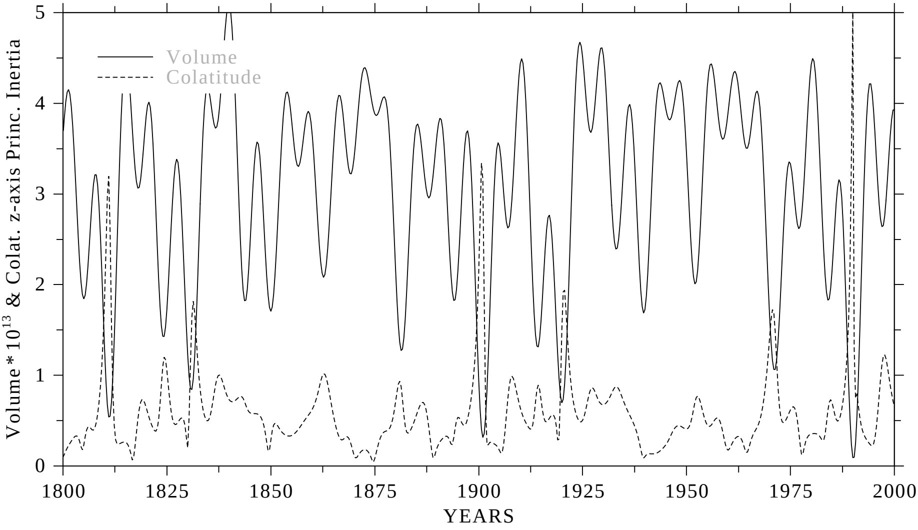

FIG. 3. Right Ascension of the Largest Principal Moment of Inertia in the Plane Perpendicular to the SSY Constant AMm Direction