NOTE 9B

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ANGULAR MOMENTA OF THE SOLAR SYSTEM THE CASE OF THE EARTH AND ITS MOON

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1. INTRODUCTION

In NOTE 9A of this little series, we assumed that the **A**ngular **M**omentu**m** (AMm) of the **S**olar **SY**stem (SSY) is constant as far as the system is conservative. This assumption is based on an axiom formulated by Euler(1775 and also 1967) for rigid bodies. This can be extended by saying that: 'the AMm of a conservative system of constant masses only subject to system internal forces is constant when, moreover, it is not subject to external torques'.

The principle of conservation of AMm (if bodies are rigid), leads to the requirement that the variable part of the AMm of the trajectory of any SSY body (sun or planet) around the barycenter of the SSY is compensated by an equal body AMm with opposite sign. We assume here that this conjecture, described in NOTE 9A, is to be accepted. The purpose of this note is then to evaluate the corresponding N-body small variable AMm and the real **T**o**RQ**ue (TRQ) which affects the luni-terrestrial system.

The first idea we exploited, is the fact that the AMm brought about by the central forces of two bodies orbiting each other is constant only when actually measured with respect to the center of mass or barycenter of these two bodies. The AMm of the orbiting bodies with respect to the inertial barycenter is not constant if the latter does not coincide with the barycenter of one of the two orbiting masses. This is what happens with our sun and all its planets separately. The case of the Earth is slightly more complex than the case of the sun, because the Earth orbits the sun which is not located at the barycenter of the SSY and the Earth further rotates around the (non-inertial) barycenter of the luni-terrestrial system. As we will see in the next section, the luni-terrestrial barycenter is more than 4700. km away from the Earth's own center of mass. All these inertially variable **A**ngular **M**oment**a** (AMa) have to be compensated by opposite body AMa inside the luni-terrestrial system.

The second idea concerns the insight that accurate ephemerides of the heliocentric orbits of the planets and the geocentric orbit of the moon are available for a reasonable past and future. Positions, velocities and accelerations of the important SSY bodies can thus be employed as an input to our problem. The advantage of having these trajectory data is that they already take the influence into account of TRQs due to third bodies influencing relative inclined orbits and TRQs due to body oblateness.

The third idea consists in getting formulae where the constant heliocentric AMa of the planets are removed. This means that we have to transform the initially barycentric AMa into heliocentric and geocentric terms. In this way we get a mixture of both variable and identifiable constant contributions to the total AMa for the different solar system planets,

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where constant parts can be eliminated. All this improves the numerical conditioning of the problem and provides direct access to the interpretation of the origin of particular AMm terms.
The theoretical concept just described should be tested by applying it to Chandler's Wobble. Unfortunately, the author has to radically descope his further projects due to severe health problems, as already mentioned in NOTE9A. We will not go beyond the formulation of the basic theory, making use of the notations introduced in NOTE9A.

2. N-BODY TRQs ACTING ON THE EARTH-MOON SYSTEM

2.1 The Geometry of the Problem

To start with, we assume that all vectors introduced hereafter, are expressed in ecliptic coordinates whose coordinate axes have to be parallel. Only the origin varies from SSY barycenter, to sun center and ends with the 'local' Earth-moon barycenter $\mathrm{B_{em}}$. Concerning notations, we will identify the sun by the subscript 's', for the moon we take the subscript 'm', we further employ '1' for Mercury, '2' for Venus and so on, up to 8 for Neptune. To denote Earth's position we will thus use $\mathbf{r_3}$ for the vector connecting the barycenter of the SSY with the center of mass of the luni-terrestrial system. The vector $\mathbf{r_{3,h}}$ is the corresponding heliocentric vector. The mass of the isolated terrestrial body is represented by M_e, while M_3 represents the mass of the Earth-moon system. Hence, we have $M_3 = M_e + M_m$. In the present context we also consider $\mathbf{u_e}$ which is the local barycentric vector pointing to the geocenter and $\mathbf{u_m}$ the equivalent vector for the moon, as shown in Fig. 1. The vectors

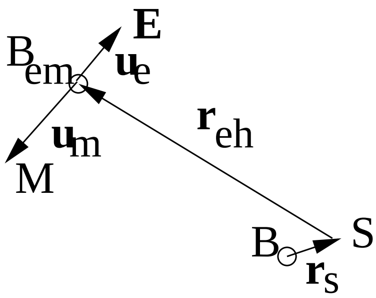

Figure 3

$\mathbf{r_e}$ and $\mathbf{r_m}$ connect the barycenter of the SSY directly with the center of mass of the terrestrial and lunar bodies, respectively. The latter two vectors are not added to Fig. 1, but they are vectorially equal to:

$$\mathbf{r_{e/m}} \;=\; \mathbf{r_s} + \mathbf{r_{3,h}} + \mathbf{u_{e/m}} \tag{1}$$

The SSY barycentric AMm $\mathbf{D_{e/m}}$ and TRQ $\mathbf{\dot{D}_{e/m}}$ of Earth and moon separately, each considered as single point masses, is then equal to:

$$\mathbf{D_{e/m}} \;=\; M_{e/m}\,(\mathbf{r_{e/m}} \times \mathbf{\dot{r}_{e/m}}) \qquad \text{and} \qquad \mathbf{\dot{D}_{e/m}} \;=\; M_{e/m}\,(\mathbf{r_{e/m}} \times \mathbf{\ddot{r}_{e/m}}) \tag{2}$$

Let $\mathbf{B_{e/m}}$ be the body attitude AMm perturbation of the Earth or the moon separately, then, according to our conjecture, it is defined by the time integral of :

$$\mathbf{\dot{B}_{e/m}} \;=\; -\,\mathbf{\dot{D}_{e/m}} \tag{3}$$

to get rid of the constant term hidden in $\mathbf{D_{e/m}}$. But the constant terms are known and their elimination is a matter of adequate basic algebra. If the TRQs generate dissipation in the non-rigid parts of the Earth and the moon, obviously only a fraction of the expected body perturbation AMa will be realized.

2.1 Elimination of Constant Angular Momenta

Based on (1) we can rewrite (2) as follows

$$\begin{aligned}\frac{\mathbf{D_{e/m}}}{M_{e/m}} &= (\mathbf{r_s} + \mathbf{r_{3,h}} + \mathbf{u_{e/m}}) \times (\dot{\mathbf{r}}_\mathbf{s} + \dot{\mathbf{r}}_\mathbf{3,h} + \dot{\mathbf{u}}_\mathbf{e/m}) \\ &= (\mathbf{r_s} \times \dot{\mathbf{r}}_\mathbf{s}) + \mathbf{r_s} \times (\dot{\mathbf{r}}_\mathbf{3,h} + \dot{\mathbf{u}}_\mathbf{e/m}) + (\mathbf{r_{3,h}} + \mathbf{u_{e/m}}) \times \dot{\mathbf{r}}_\mathbf{s} + \\ &\quad (\mathbf{r_{3,h}} + \mathbf{u_{e/m}}) \times (\dot{\mathbf{r}}_\mathbf{3,h} + \dot{\mathbf{u}}_\mathbf{e/m}) \end{aligned} \tag{4}$$

stressing the important role of $\mathbf{r_s}$. In section 2 of NOTE9A, we have derived that

$$\mathbf{r_s} = -\sum_{i=1}^{8} m_i \, \mathbf{r_{i,h}} \tag{5}$$

where m_i is the shorthand for M_i/M_{tot} with the total SSY mass M_{tot} defined by

$$M_{tot} = M_s + \sum_{i=1}^{8} M_i \tag{6}$$

By substituting (5) into (4), the first vector product on the right hand side of (4) is now developed with the purpose to introduce a further shorthand. We get

$$\begin{aligned}(\mathbf{r_s} \times \dot{\mathbf{r}}_\mathbf{s}) &= \left(-\sum_{i=1}^{8} m_i \, \mathbf{r_{i,h}}\right) \times \left(-\sum_{j=1}^{8} m_j \, \dot{\mathbf{r}}_\mathbf{j,h}\right) \\ &= \sum_{i=1}^{7} \sum_{j=i+1}^{8} m_i \, m_j \, (\mathbf{r_{i,h}} \times \dot{\mathbf{r}}_\mathbf{j,h} + \mathbf{r_{j,h}} \times \dot{\mathbf{r}}_\mathbf{i,h}) + \\ &\quad \sum_{i=1}^{8} m_i^2 \, \mathbf{r_{i,h}} \times \dot{\mathbf{r}}_\mathbf{i,h} \end{aligned} \tag{7}$$

This important abbreviation is namely:

$$\mathbf{p_{jk}} = m_j \, m_k \, (\mathbf{r_{j,h}} \times \dot{\mathbf{r}}_\mathbf{k,h} + \mathbf{r_{k,h}} \times \dot{\mathbf{r}}_\mathbf{j,h}) \tag{8}$$

which we call the **M**i**X**ed **M**omenta (MXM) of the bodies j and k, with $\mathbf{p_{jk}} = \mathbf{p_{kj}}$. The constant terms $\mathbf{p_{kk}} = m_k^2 \, \mathbf{r_{k,h}} \times \dot{\mathbf{r}}_\mathbf{k,h}$ will be systematically ignored in all what follows. To all this we also add the complementary abbreviation:

$$\mathbf{p_{i,e/m}} = m_i \, m_{e/m} \, (\mathbf{r_{i,h}} \times \dot{\mathbf{u}}_\mathbf{e/m} + \mathbf{u_{e/m}} \times \dot{\mathbf{r}}_\mathbf{i,h}) \tag{9}$$

with $m_{e/m} = M_{e/m}/M_{tot}$. Further, the AMa $M_{e/m}(\mathbf{u_{e/m}} \times \dot{\mathbf{u}}_{\mathbf{e/m}})$ cannot be considered to be constant.
Let us return to (4) and the expansion of its variable vector-terms. We obtain:

$$\mathbf{D_{e/m}} \;=\; M_{e/m} \sum_{i=1}^{7} \sum_{j=i+1}^{8} \mathbf{p_{ij}} \tag{10a}$$

$$-\; M_{tot}\,(\sum_{i\neq 3}^{8} \mathbf{p_{i3}} \;+\; \sum_{i=1}^{8} \mathbf{p_{i,e/m}}) \tag{10b}$$

$$+\; M_{e/m}\,(\mathbf{r_{3,h}} \times \dot{\mathbf{u}}_{\mathbf{e/m}} \;+\; \mathbf{u_{e/m}} \times \dot{\mathbf{r}}_{\mathbf{3,h}} \;+\; \mathbf{u_{e/m}} \times \dot{\mathbf{u}}_{\mathbf{e/m}}) \tag{10c}$$

We notice that the coefficients on line (10a) have the form $M_{e/m}\, M_k\, M_j/M_{tot}^2$, while similar coefficients on line (10b) are equal to $M_{e/m}\, M_j/M_{tot}$. Dividing the former by the latter yields M_k/M_{tot}. Employing the mass values given in the appendix, with $M_{tot} = 332717.6$ Earth masses and selecting Jupiter and Saturn for M_j and M_k , the ratio just mentioned is equal to $317.9/332717.2 = 0.9555\,10^{-3}$. It means that the coefficients of the separate terms of line (10a) are at least thousand times smaller than those of line (10b). Nevertheless, the mixed momenta correspond to vectors whose magnitude oscillates around zero and thereby small terms still contribute to accuracy when being close to either zero AMa or TRQs.

2.3 Geocentric Moon Ephemerides and $\mathbf{u_{e/m}}$ Dependencies

By assuming that the moon is at 384400. km from the Earth center, we get an offset of the Earth's center of mass from the local barycenter equal to 4787 km. This is more than 70% of the equatorial Earth radius. In a similar scenario, the equivalent offset of the local barycenter of Saturn is at most 200. km and in the case of Jupiter we get only half that value, thus negligible when considering the sizes of these latter giant planets. The local barycentric offset of the Earth center is the largest in the solar system when excluding the sun. This is a more than sufficient reason to include the moon in our analysis.
In this subsection we start by replacing $\mathbf{u_{e/m}}$ by vectors proportional to the lunar geocentric position in ecliptic co-ordinates, which we represent by $\mathbf{r_{m,g}}$. Therefore, we first observe that the center of mass, $\mathrm{B_{em}}$ in Fig.1, implies

$$M_e\,\mathbf{u_e} \;+\; M_m\,\mathbf{u_m} \;=\; \mathbf{0} \tag{11}$$

and together with $\mathbf{u_m} \;-\; \mathbf{u_e} \;=\; \mathbf{r_{m,g}}$ this leads to:

$$\mathbf{u_m} \;=\; \frac{M_e}{M_e \;+\; M_m}\,\mathbf{r_{m,g}}, \qquad \mathbf{u_e} \;=\; -\,\frac{M_m}{M_e \;+\; M_m}\,\mathbf{r_{m,g}} \tag{12}$$

where the difference between the signs of $\mathbf{u_m}$ and $\mathbf{u_e}$ puts an end at the e/m parallel formulae maintained up to (10). This necessary splitting affects all terms of (10) where vectors with the subscript (e/m) appear and whose sum will be represented by $\mathbf{K_e}$ and $\mathbf{K_m}$, except for $M_{e/m}(\mathbf{u_{e/m}} \times \dot{\mathbf{u}}_{\mathbf{e/m}})$ which we will write $\mathbf{H_e}$ and $\mathbf{H_m}$.
For $\mathbf{K_e}$ and $\mathbf{K_m}$ we obtain:

$$\mathbf{K_e} \;=\; +\,\frac{M_m}{M_e \;+\; M_m}\left(\,M_{tot}\sum_{i=1}^{8}\mathbf{p_{i,g}} - \,M_e\,(\mathbf{r_{3,h}} \times \dot{\mathbf{r}}_{\mathbf{m,g}} \;+\; \mathbf{r_{m,g}} \times \dot{\mathbf{r}}_{\mathbf{3,h}})\right) \tag{13}$$

and

$$\mathbf{K_m} \;=\; -\,\frac{M_e}{M_e\,+\,M_m}\left(M_{tot}\sum_{i=1}^{8}\mathbf{p_{i,g}} + \,M_m\,(\mathbf{r_{3,h}}\times\dot{\mathbf{r}}_{\mathbf{m,g}}\,+\,\mathbf{r_{m,g}}\times\dot{\mathbf{r}}_{\mathbf{3,h}})\right) \tag{14}$$

For $\mathbf{H_m}$ and $\mathbf{H_e}$ we further find

$$\mathbf{H_m} = M_m(\mathbf{u_m}\times\dot{\mathbf{u}}_\mathbf{m}) = \frac{M_m\,M_e^2}{(M_e+M_m)^2}\,\mathbf{r_{m,g}}\times\dot{\mathbf{r}}_{\mathbf{m,g}}$$

$$\mathbf{H_e} = M_e\,(\mathbf{u_e}\,\times\,\dot{\mathbf{u}}_\mathbf{e}) = \;\frac{M_e\,M_m^2}{(M_e+M_m)^2}\,\mathbf{r_{m,g}}\times\dot{\mathbf{r}}_{\mathbf{m,g}}$$

$$|\mathbf{H_e}|/|\mathbf{H_m}| = M_m/M_e, \qquad \mathbf{H_e}+\mathbf{H_m} = \frac{M_eM_m}{(M_e+M_m)}\,\mathbf{r_{m,g}}\times\dot{\mathbf{r}}_{\mathbf{m,g}} \tag{15}$$

If we have included $\mathbf{u_{e/m}}\times\dot{\mathbf{u}}_\mathbf{e/m}$ as a variable term in the two previous equations, it is because the moon orbit is also subjected to a non-negligible influence of solar gravitation. This can be deduced from the well known facts that the line of nodes of the moon orbit in the ecliptic regresses by some 20^0 per year. Moreover, its apsidal line makes a full revolution in 8.85 yr. These are sufficient reasons to assume that the lunar orbit around the Earth does not satisfy Kepler's second law to a sufficient approximation. But in this respect we realize that $\mathbf{H_e}$ and $\mathbf{H_m}$ are well known precise numerical entities which are not directly linked to the solar barycentric offset. The corresponding AMm mass point deviation from its constant mean is in principle accurately known. Only the sign of the attitude body impact depends on the acceptance or refusal of the mass point-body AMm conjecture. We have not checked the availability of relevant potential literature and did not pursue this track.

3. ANGULAR MOMENTUM ANALYSIS

3.1 Periodicities

In section 5 of NOTE 9A it has been shown that the ecliptic z-component of the MXM have an approximate periodic behavior proportional to $\sin(|\omega_k - \omega_\ell|\,t)$ where t is time in yr, $\omega_j = 2\pi/T_j$ is the rotation rate component of planet j for which T_j is the orbital period. Consequently, the combined period $T_{k\ell}$ is defined by

$$\omega_{k\ell} \;=\; \frac{2\pi}{T_{k\ell}} \;=|\,\frac{2\pi}{T_k}-\frac{2\pi}{T_\ell}\,| \tag{15}$$

and known as the synodic period separating either successive conjunctions or oppositions of the planets k and ℓ.

In this section we first look for the fundamental frequencies linked to the MXM $\mathbf{p_{3k}}$ which appear in the left term of (10b) and thus apply to Earth and moon alike. They all have a mean periodicity exceeding a terrestrial year, as shown in table 1. Other, thousand times smaller contributions are coming from (14a). They can be limited to the MXM consisting of the combination of two Jovian planets. All of these periods exceed the Jupiter sidereal orbital period $T_5 = 11.862$yr with the lowest value $T_{58} = 12.782$yr up to $T_{78} = 171.41$yr.

Table 1. Earth-Planet Synodic Periods In Years and Corresponding Earth Spin Axis Rotation Component Periods in days (in brackets)

VENUS	MARS	JUPITER
1.597(636.22)	2.135(850.37)	1.092(434.95)
SATURN	URANUS	NEPTUNE
1.035(412.28)	1.012(403.08)	1.006(400.72)

Now, reminding the latitude dependency of the rotation period of the oscillation plane of Foucault's pendulum, the time dependency of $\mathbf{p_{3k}}$ of the z-component of the AMm of planet k in ecliptic co-ordinates is heuristically like the Earth spin rate at the pole for the pendulum. The terrestrial spin axis is at 24.35^0 away from the ecliptic z-axis. The z-component of any $\mathbf{p_{3k}}$ carries almost the full magnitude of the MXM. This AMm direction can, in a good approximation, be interpreted as coincident with a rotation rate with a period T_{3k}. Consequently, only the projection of this rotation vector onto the Earth 'spin axis' is translated into an Earth spin effect. This effect is thus reduced to the projection $\cos(24.35^0)\,\omega_{3k}$ and the corresponding apparent synodic periods felt on Earth are equal to $S_{3k} = T_{3k}/\cos(24.35^0)$. The latter values expressed in days are shown between brackets in table 1. There we notice that the apparent synodic periods of the Jovian planets Jupiter and Saturn fit into the interval of approximate, observed Chandler Wobble periods.

Let us now Consider the variable MXM terms in (10) where the moon and Earth are involved as $\mathbf{u_e}$ and/or $\mathbf{u_m}$. By (12) these $\mathbf{u}$-vectors are subject to the orbital period of the lunar geocentric position vector. Consequently, synodic periods between the moon or Earth and planet k represented on the right of (10b), are comprised in a narrow interval defined by $T_m = 27.322\,\text{days} < T_{mk} \leq T_{m3} = 29.531\,\text{days}$, as one can easily verify. The same applies 'a fortiori' to the MXM contained in (10c). Thus moon and Earth are both separately affected by disturbance AMa and corresponding TRQs whose period are all very close to the moon's sidereal orbital period.

3.2 Magnitudes

Excluding Mercury, we consider 7 planets. Consequently, we have 21 different MXM $\mathbf{p_{ik}}$ vectors. To get a feel for the orders of magnitude involved, we have tabulated the different magnitudes c_{ik} of the mixed vector products $\mathbf{c_{ik}} = \mathbf{c_{ki}} = \mathbf{r_{i,h}} \times \dot{\mathbf{r}}_{\mathbf{k,h}} + \mathbf{r_{k,h}} \times \dot{\mathbf{r}}_{\mathbf{i,h}}$. We considered the sums of c_{ik} over 81 years, sampling one set of values once a month. The units employed are AU for length and day for time. It happens that the sums obtained are all quite similar. This is best exemplified by giving the sums obtained for Venus successively combined with the other planets and the same for the Earth. We got:

$\Sigma c_{23} = 20.62$, $\Sigma c_{24} = 25.47$, $\Sigma c_{25} = 55.57$, $\Sigma c_{26} = 100.3$, $\Sigma c_{27} = 199.4$, $\Sigma c_{28} = 314.5$.

For the Earth we obtained:

$\Sigma c_{32} = 20.62$, $\Sigma c_{34} = 24.95$, $\Sigma c_{35} = 60.11$, $\Sigma c_{36} = 104.9$, $\Sigma c_{37} = 207.2$, $\Sigma c_{38} = 322.7$.

Also all other mixed vector products satisfy the inequality $\Sigma c_{23} \leq \Sigma c_{ij} \leq \Sigma c_{38}$ for $2 \leq i \neq j \leq 8$.

The largest extremum of each mixed vector product met in 81 years were identified and again similar results were obtained. The location of the boundaries are the same as before, for we found $\sup(c_{23}) = 0.0328 \leq \sup(c_{ij}) \leq \sup(c_{38}) = 0.7400$.

Hence, the amplitudes of the MXM are ruled by the different mass combinations inside the coefficients of (10). Therefore, our attention goes to terms $M_{tot}\mathbf{p_{i3}}$ in (10b) which we consider to be responsible for the bulk of the long(er) periodic minute perturbations of the Earth attitude. Coefficients involved are $M(3,i) = M_3\, M_i/M_{tot}$ for which we obtain: $M(3,2)$ = .248E-5, $M(3,4)$ = .326E-6, $M(3,5)$ = .968E-3, $M(3,6)$ = .290E-3, $M(3,7)$ = .443E-4, and $M(3,8)$ = .524E-4. This corresponds to an interval $M(3,4)$ = .326E-6$\leq M(3,k) \leq M(3,5)$ = .968E-3. The comparison with the coefficients $M(3,j,k)$ applicable to (10a) reveals that the highest boundary of $M(3,j,k)$ corresponds more or less to the lowest boundary of the $M(3,k)$ coefficients, or $M(3,5,6) \approx M(3,4)$. Hence, $M_{tot}\sum_{i\neq 3}\mathbf{p_{i3}}$ is by and large at least responsible for the first three significant digits of the AMm and TRQ perturbations with periods longer than one year. This may be sufficient for plotting purposes, but higher precision requires the inclusion of the AMm vectors involved in (10a) as well.

4. CONCLUDING REMARKS

At this point I have reached the task of predicting the Earth body response to the perturbing TRQs presented before. Theoretically one should integrate the Euler differential equations for the rotation vector (=spin vector) of the terrestrial body subject to the sum of TRQs acting on it. The question arises how one deals with the TRQ exerted by the moon on the oblate Earth combined or not in the integration of the perturbing planetary TRQs. One can even imagine some simpler approximations with uncertain accuracy.
I know that there are enough clever and experienced scientists able to tackle this problem, and this also without my thoughts which would be void of any practical verification. But exactly this verification is actually the point. At any rate, it is better to present a note which is missing this last step, than to take the risk to present no note at all in the end.

Appendix

Orbital Parameters of the Moon and Major Planets

	a(AU)	e	M(EM)	Sidereal Period(yr)	Inclination (deg)
Moon*	384400.*	0.055	0.0123	27.322*	$5^0\,9'$
Venus	0.723	0.007	0.815	0.615	$3^0\,23.7'$
Earth	1.000	0.017	1.000	1.000	–
Mars	1.524	0.093	0.107	1.881	$1^0\,51.0'$
Jupiter	5.203	0.048	317.9	11.862	$1^0\,18.3'$
Saturn	9.539	0.055	95.18	29.458	$2^0\,29.3'$
Uranus	19.191	0.047	14.54	84.015	$0^0\,46.3'$
Neptune	30.061	0.010	17.13	164.79	$1^0\,46.3'$

* for the moon the unit of length is km and the unit of time is given in days

REFERENCES

Relevant references are contained in note 9A.